ANDERSON’S THEOREM FOR COMPACT OPERATORS

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Abstract. It is shown that if $A$ is a compact operator on a Hilbert space with its numerical range $W(A)$ contained in the closed unit disc $\overline{D}$ and with $W(A)$ intersecting the unit circle at infinitely many points, then $W(A)$ is equal to $\overline{D}$. This is an infinite-dimensional analogue of a result of Anderson for finite matrices.

The numerical range $W(A)$ of a bounded linear operator $A$ on a complex Hilbert space $H$ is the subset $\{\langle Ax, x \rangle : x \in H, \|x\| = 1\}$ of the complex plane, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the inner product and norm in $H$, respectively. Basic properties of the numerical range can be found in [5, Chapter 22] or [4].

In the early 1970s, Joel Anderson proved an interesting result on the numerical ranges of finite matrices. Namely, if $A$ is an $n$-by-$n$ complex matrix, considered as an operator on $\mathbb{C}^n$ equipped with the standard inner product and norm, with its numerical range $W(A)$ contained in the closed unit disc $\overline{D}$ ($\overline{D} \equiv \{z \in \mathbb{C} : |z| < 1\}$) and intersecting the unit circle $\partial \overline{D}$ at more than $n$ points, then $W(A) = \overline{D}$ (cf. [9, p. 507]). The purpose of this paper is to prove an infinite-dimensional analogue of Anderson’s result for compact operators.

Theorem 1. If $A$ is a compact operator on a Hilbert space with $W(A)$ contained in $\overline{D}$ and $W(A)$ intersecting $\partial \overline{D}$ at infinitely many points, then $W(A) = \overline{D}$.

Anderson never published his proof of the above-mentioned result. As related by him many years later via an e-mail to the second author, his proof was based on the application of Bézout’s theorem to the Kippenhahn curve of the matrix $A$. Generalizations of this result along this line can be found in [3]. In recent years, there appeared three more proofs. One is by Dritschel and Woerdeman [2, Theorem 5.8], based on the canonical decomposition and radial tuples for numerical contractions developed by them. (A numerical contraction is an operator $A$ with $W(A) \subseteq \overline{D}$.) The second one is due to the second author (cf. [12, Lemma 6]); it depends on the classical Riesz-Fejér theorem on nonnegative trigonometric polynomials. More recently, Hung gave another proof in his Ph.D. dissertation [6, Theorem 4.2.1] by making use of Ando’s characterization of numerical contractions.

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Figure 2.

We will prove Theorem 1 using the support function \( d_A \) of the compact convex set \( W(A) \) of an operator \( A \):

\[
    d_A(\theta) = \max_{W} \{ \Re(e^{-i\theta}A) \} \\
    = \max_{W} \{ \cos \theta \Re A + \sin \theta \Im A \}
\]

for \( \theta \) in \( \mathbb{R} \), where \( \Re A = (A + A^*)/2 \) and \( \Im A = (A - A^*)/(2i) \) are the real and imaginary parts of \( A \). Note that \( d_A(\theta) \) is simply the signed distance from the origin to the supporting line \( L_\theta \) of \( W(A) \) which is perpendicular to the ray \( R_\theta \) from the origin that forms angle \( \theta \) from the positive \( x \)-axis (cf. Figure 2).

Our main tool is the next theorem, due to Rellich [10, p. 57], on the analytic perturbation for multiple eigenvalues of Hermitian operators; an elegant and elementary proof can be found in [11, p. 376]. The present form is from [8, Theorem 3.3].

**Theorem 3.** Let \( \theta \mapsto A_\theta \) be a real analytic function from an open interval \( I \) of \( \mathbb{R} \) to Hermitian operators on a fixed Hilbert space, and let \( d(\theta) = \max_{W} \{ A_\theta \} \) for \( \theta \) in \( I \). Assume that for some \( \theta_0 \) in \( I \), \( d(\theta_0) \) is an isolated eigenvalue of \( A_{\theta_0} \) with finite multiplicity \( n \). Then there is an open subinterval \( J \) of \( I \) which contains \( \theta_0 \) and there are \( m, 1 \leq m \leq n \), real analytic functions \( d_1, \ldots, d_m : J \to \mathbb{R} \) such that

(a) \( d_1(\theta_0) = \cdots = d_m(\theta_0) = d(\theta_0) \),

(b) for every \( \theta \) in \( J \setminus \{ \theta_0 \} \), the \( d_j(\theta) \)'s are distinct isolated eigenvalues of \( A_\theta \) with respective multiplicity \( n_j \) independent of \( \theta \) which satisfies \( \sum_{j=1}^m n_j = n \),

(c) there is some \( d_{j_1} \) (resp., \( d_{j_2} \)) such that \( d(\theta) = d_{j_1}(\theta) \) (resp., \( d(\theta) = d_{j_2}(\theta) \)) for all \( \theta \), \( \theta < \theta_0 \) (resp., \( \theta > \theta_0 \)) in \( J \), and

(d) \( d(\theta) = \max\{d_1(\theta), \ldots, d_m(\theta)\} \) for all \( \theta \) in \( J \).

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** We first express our assumptions in terms of \( d_A \). The condition \( W(A) \subseteq \mathbb{B} \) is obviously equivalent to \( d_A(\theta) \leq 1 \) for all \( \theta \). Under this, we
then have, for a fixed \( \theta \), the equivalence of \( e^{i\theta} \in \overline{W(A)} \) and \( d_A(\theta) = 1 \). Indeed, \( e^{i\theta} \) belonging to \( \overline{W(A)} \) is equivalent to 1 belonging to \( \overline{W(e^{-i\theta}A)} \), which is the same as 1 belonging to \( \overline{W(e^{-i\theta}A)} = \overline{W(\text{Re}(e^{-i\theta}A))} \) (because \( W(e^{-i\theta}A) \subseteq \mathbb{D} \)) or \( d_A(\theta) = 1 \).

Now let \( e^{i\theta_n}, n \geq 1, \theta_n \in [0, 2\pi] \), be a sequence of distinct points in \( \overline{W(A)} \cap \partial \mathbb{D} \). Passing to a subsequence, we may assume that \( \theta_n \) converges to \( \theta_0 \in [0, 2\pi] \). Since \( d_A(\theta_n) = 1 \) for all \( n \) and the function \( \theta \mapsto \overline{W(\text{Re}(e^{-i\theta}A))} \) is continuous (cf. [5, Solution 220]), we obtain \( d_A(\theta_0) = 1 \). Moreover, since \( \overline{W(\text{Re}(e^{-i\theta_0}A))} \) equals the convex hull of the spectrum of the compact operator \( \text{Re}(e^{-i\theta_0}A) \), we infer that \( d_A(\theta_0) \) is an isolated eigenvalue of \( \text{Re}(e^{-i\theta_0}A) \) with finite multiplicity. Thus Theorem 3 may be applied to obtain two real analytic functions \( d_1 \) and \( d_2 \) on some neighborhood \( J = (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \) of \( \theta_0 \) such that \( d_A = d_1 \) on \( (\theta_0 - \varepsilon, \theta_0] \) and \( d_A = d_2 \) on \( [\theta_0, \theta_0 + \varepsilon) \). Without loss of generality, we may assume that \( (\theta_0 - \varepsilon, \theta_0] \) contains infinitely many \( \theta_n \)’s. Hence \( d_1(\theta_n) = d_A(\theta_n) = 1 \) for such \( \theta_n \)’s. Since \( \theta_n \) converges to \( \theta_0 \) and \( d_1 \) is analytic on \( J \), we obtain \( d_1 = 1 \) on \( J \). Therefore, \( d_1 \leq d_A \leq 1 \) implies that \( d_A = 1 \) on \( J \). Let \( \alpha = \{ \theta \in \mathbb{R} : d_A(\theta) = 1 \} \). The above arguments also show that if \( \theta' \) is a limit point of \( \alpha \), then there is some neighborhood \( \theta' - \varepsilon', \theta' + \varepsilon' \) contained in \( \alpha \). Now let \( a = \sup \{ \theta \in \mathbb{R} : [\theta_0, \theta] \subseteq \alpha \} \) and \( b = \inf \{ \theta \in \mathbb{R} : (\theta, \theta_0] \subseteq \alpha \} \). We infer from the above that \( a = \infty \) and \( b = -\infty \), that is, \( \alpha = \mathbb{R} \). This shows that \( d_A = 1 \) on \( \mathbb{R} \) or, equivalently, \( \partial \mathbb{D} \subseteq \overline{W(A)} \). As we have seen in the first paragraph of this proof, \( d_A(\theta) = 1 \) is equivalent to \( 1 \in \overline{W(\text{Re}(e^{-i\theta}A))} \). Since this latter set equals the convex hull of the spectrum of the compact operator \( \text{Re}(e^{-i\theta}A) \), we infer that 1 is an eigenvalue of \( \text{Re}(e^{-i\theta}A) \). Hence 1 is in \( \overline{W(\text{Re}(e^{-i\theta}A))} \) or in \( W(e^{-i\theta}A) \) (since \( W(e^{-i\theta}A) \subseteq \mathbb{D} \)), which is the same as \( e^{i\theta} \) in \( W(A) \). We conclude that \( \partial \mathbb{D} \subseteq \overline{W(A)} \). The convexity of \( W(A) \) then implies that \( W(A) = \overline{\mathbb{D}} \), completing the proof.

An alternative proof for the last part of the preceding proof is, after obtaining \( \overline{W(A)} = \overline{\mathbb{D}} \) from \( \partial \mathbb{D} \subseteq \overline{W(A)} \) and the convexity of \( \overline{W(A)} \), to invoke [5] Solution 213 that any compact operator \( A \) with \( \theta \in W(A) \) has \( W(A) \) closed, concluding that \( W(A) = \overline{\mathbb{D}} \).

We end this paper with some further remarks. First, any compact operator \( A \) with \( W(A) = \overline{\mathbb{D}} \) must have norm bigger than one. This is because if \( \|A\| \leq 1 \), then from the equality case of the Cauchy-Schwarz inequality, we easily derive that \( W(A) \cap \partial \mathbb{D} = \partial \mathbb{D} \) consists of eigenvalues of \( A \), which is impossible for the compact \( A \). Second, we note that in Theorem 1 the condition that \( \overline{W(A)} \) intersects \( \partial \mathbb{D} \) at infinitely many points cannot be weakened. For example, for each \( n \geq 1 \), if \( A_n \) is the finite-rank operator \( \text{diag}(1, \omega_n, \ldots, \omega_n^{n-1}, 0, 0, \ldots) \), where \( \omega_n \) is the \( n \)th primitive root of 1, then \( W(A_n) \subseteq \overline{\mathbb{D}} \) and \( \overline{W(A_n)} \) intersects \( \partial \mathbb{D} \) at the \( n \) points \( 1, \omega_n, \ldots, \omega_n^{n-1} \). Finally, Theorem 1 can be generalized from the unit disc to any elliptic disc centered at the origin: if \( A \) is a compact operator with \( W(A) \) contained in the closed elliptic disc

\[
E = \{ x + iy \in \mathbb{C} : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \}, \quad a, b > 0,
\]
and with \( \overline{W(A)} \) intersecting \( \partial E \) at infinitely many points, then \( W(A) = E \). This can be reduced to Theorem 1 by considering the affine transform
\[
B = \frac{1}{a} \text{Re} A + \frac{i}{b} \text{Im} A
\]
of \( A \) since the numerical range of \( B \) equals \( \mathbb{D} \).

[7] and [1] are the other papers which contain information on the numerical ranges of compact operators.

References


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