GRAPHIC SOLUTION of $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$

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IN ELEMENTARY algebra work problems, water-pipe delivery problems, and similar problems are based on the relationship

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

The units for the three variables may be of any magnitude, fundamental or derived. The only requirement is that all three variables be in the same units.

For example, if a job can be finished by A alone in 3 days and by B alone in 6 days and can be finished by A and B working together in r days, then r can be found by the above formula, $1/r = 1/3 + 1/6$. Here, the uniform units used are days. However, if we know that the job can be done by A alone in 3 days and by B alone in 6 days, then how long does it take B to finish it alone? Here, r and p are 2 and 3 days, respectively; q can be found by solving the equation $1/q = 1/2 - 1/3$.

Similarly, suppose a tank of water can be emptied in 10 minutes by opening pipe A alone, in 6 minutes by opening pipe B alone, and in r minutes by opening pipes A and B together. Then, using the same formula, $1/r = 1/10 + 1/6$. Here the uniform units used are minutes.
demonstrate the close relationship between algebra and geometry. The graphic method is described and proved as follows.

Two line segments, \( \overline{AB} \) and \( \overline{CD} \) are drawn perpendicular to \( \overline{LM} \), as shown in figure 1. Segments \( \overline{AD} \) and \( \overline{BC} \) are drawn, intersecting at \( F \), and \( \overline{FE} \) is drawn perpendicular to \( \overline{LM} \). Then, as will be proved, \( 1/|EF| = 1/|AB| + 1/|CD| \), or \( 1/r = 1/p + 1/q \).

**Proof.** In triangle \( ABC \),
\[
\frac{p}{r} = \frac{AC}{EC}. \tag{1}
\]
In triangle \( CDA \),
\[
\frac{q}{r} = \frac{AC}{AC - EC}. \tag{2}
\]
From (1),
\[
EC = AC \cdot \frac{r}{p}.
\]
Substituting into (2),
\[
\frac{q}{r} = \frac{AC}{AC - AC \cdot \frac{r}{p}}
\]
or
\[
\frac{q}{r} = \frac{1}{1 - \frac{r}{p}}.
\]

Substituting \( 1 - \frac{r}{p} \) into (2),
\[
q \left( 1 - \frac{r}{p} \right) = r.
\]

Dividing by \( qr \),
\[
\frac{1}{r} \cdot \frac{1}{p} = \frac{1}{q}.
\]

When more than two men work or more than two pipes are open, the right-hand member of the equation can be extended, thus:

\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q} + \frac{1}{s} + \ldots + \frac{1}{n}.
\]

There are many physical phenomena to which this method applies. When two electrical resistances \( r_1 \) and \( r_2 \) are connected in parallel, the combined resistance \( R \) is given by \( 1/R = 1/r_1 + 1/r_2 \). Here \( r_1 \) and \( r_2 \) represent resistance in ohms. Again, when two electrical capacitors are connected in series the combined capacitance \( C \) is given by \( 1/C = 1/c_1 + 1/c_2 \), where \( c_1 \) and \( c_2 \) are individual capacitances in farads or microfarads.

In the study of spherical mirrors and lenses, the focal length \( f \) of a mirror or a lens can be found using \( 1/f = 1/d_0 + 1/d_1 \). Here \( d_0 \) is the distance of an object from the vertex of a mirror or from the center of a lens, and \( d_1 \) is the distance of the image from the vertex or from the center. The units of distance can be centimeters, meters, inches, or feet, so long as they are the same for all the variables in the formula.

A graphic method is available for the approximate solution of the equation \( 1/r = 1/p + 1/q \) when any two of the three variables are given. This method is presented here not so much to show another method of solution as to
It should be noted that the above formula is independent of the distance $AC$ between the line segments $AB$ and $CD$, therefore these segments may be drawn any distance apart, and the value of $r$ remains the same. This may be seen in figure 2, where the segment $p$ and the four different segments all of length $q$ are drawn at different distances apart, but the vertical segments $r_1, r_2, r_3, \ldots$ are the same length. This property facilitates the solution of more complicated problems, as will be shown later.

Now let us use the graphic method to solve the work problem given at the beginning of this article. Use a convenient unit to represent 1 day. As shown in figure 3, draw two line segments of lengths $AB=3$ and $CD=6$, both perpendicular to $LM$ (sec fig. 4). Draw $AF$ and $BF$. $BF$ meets $LM$ at $C$. Erect a perpendicular to $LM$ at $C$, intersecting $AF$ at $D$. Then $CD$ is the required line segment; it measures 6 units, indicating 6 days, the time required by $B$ to finish the job when working alone.

Similarly, water-pipe problems and problems involving resistances in parallel, inductances in parallel, and capacitances in series can all be worked out by the graphic method. No numerical example is necessary for any one of these problems, because they are all done in the same way as work problems. The graphic method is also helpful when there are more than two men working, more than two resistances in parallel, and so on. Suppose there are four resistances of 20, 60, 85, and 140 ohms in parallel; the total resistance, $R$, is:

\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q}
\]

or

\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q}
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solution of more complicated forms of water-tank problems. Suppose a bathtub can be filled in 10 minutes by the hot-water faucet alone and in 6 minutes by the cold-water faucet alone. When the tub is full, it can be emptied in 5 minutes by the drainpipe. If both faucets and the drainpipe are open, can the tub be filled? If so, how long does it take?

By formula, \( \frac{1}{r} = \frac{1}{10} + \frac{1}{6} - \frac{1}{5} \), or \( \frac{1}{r} = \frac{1}{15} \), and \( r = 15 \) minutes.

Figure 6 is the graphic solution of the same problem. The construction is self evident, and no explanation of procedure is necessary. The line \( JK \) is the line segment required; its length above the line \( LM \) indicates that the tub is filled in 15 minutes, the result as obtained by calculation.

It may be noted that this problem could have been solved equally well by various other pairings. For example, \( a \) might have been paired with \( b \) to obtain \( e \), then \( c \) with \( d \) to obtain \( f \), then \( f \) with \( e \) to obtain \( g \). In fact, any possible pairing can be used.

If negative values are introduced, the use of the formula and its graphic solution can be extended to the case where negative values are allowed.