On the Spanning $w$-Wide Diameter of the Star Graph

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Let $u$ and $v$ be any two distinct nodes of an undirected graph $G$, which is $k$-connected. A container $C(u,v)$ between $u$ and $v$ is a set of internally disjoint paths $\{P_1, P_2, \ldots, P_w\}$ between $u$ and $v$ where $1 \leq w \leq k$. The width of $C(u,v)$ is $w$ and the length of $C(u,v)$ (written as $l(C(u,v))$) is $\max\{l(P_i) \mid 1 \leq i \leq w\}$. A $w$-container $C(u,v)$ is a container with width $w$. The $w$-wide distance between $u$ and $v$, $d_w(u,v)$, is $\min\{l(C(u,v)) \mid C(u,v)$ is a $w$-container$. A $w$-container $C(u,v)$ of the graph $G$ is a $w^*$-container if every node of $G$ is incident with a path in $C(u,v)$. That means that the $w$-container $C(u,v)$ spans the whole graph. Let $S_n$ be the $n$-dimensional star graph with $n \geq 5$. It is known that $S_n$ is bipartite. In this article, we show that, for any pair of distinct nodes $u$ and $v$ in different partite sets of $S_n$, there exists an $(n-1)^*$-container $C(u,v)$ and the $(n-1)$-wide distance $d_{n-1}(u,v)$ is less than or equal to $\frac{n^2-1}{n-2}+1$. In addition, we also show the existence of a $2^*$-container $C(u,v)$ and the 2-wide distance $d_2(u,v)$ is bounded above by $\frac{n^2}{2} + 1$.

Keywords: diameter; hamiltonian; hamiltonian laceable; star graphs

1. BASIC DEFINITIONS

An interconnection network connects the processors of parallel computers. Its architecture can be represented as a graph, in which the nodes correspond to processors and the edges correspond to connections. Hence, we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology for computer networks. The $n$-cube is one of the most popular topologies [17]. The $n$-dimensional star network $S_n$ was proposed in [1] as “an attractive alternative to the $n$-cube” topology for interconnecting processors in parallel computers. Since its introduction, the network $S_n$ has received considerable attention. Akers and Krishnamurthy [1] showed that the star graphs are node transitive and edge transitive. Jwo et al. [15] showed that the star graphs are bipartite. Star graphs are able to embed grids [15]: trees [3, 5, 8], and hypercubes [22]. Cycle embeddings and path embeddings are studied in [10–13, 15, 18, 23]. The diameter and fault diameters of star graphs were computed in [1, 16, 24]. Some interesting properties of star graphs are studied in [7, 9, 19].

For graph definitions and notation we follow [4]. $G = (V,E)$ is a graph if $V$ is a finite set and $E$ is a subset of $\{(u,v) \mid (u,v)$ is an unordered pair of $V\}$. We say that $V$ is the node set and $E$ is the edge set. A graph $G$ is vertex transitive if there is an isomorphism $f$ from $G$ into itself such that $f(u) = v$ for any two nodes $u$ and $v$ of $G$. A graph $G$ is edge transitive if there is an isomorphism $f$ from $G$ into itself such that $f((u,v)) = (x,y)$ for any two edges $(u,v)$ and $(x,y)$ of $G$. For a node $u$ in graph $G$, $N_G(u)$ denotes the neighborhood of $u$, which is the set $\{v \mid (u,v) \in E\}$. For any node $u$ of $V$, we denote the degree of $u$ by $\deg_G(u) = |N_G(u)|$. 

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A graph $G$ is $k$-regular if $deg_G(u) = k$ for all nodes $u$ in $G$. Two nodes $u$ and $v$ are adjacent if $(u, v) \in E$. A path $P$ is a sequence of adjacent nodes, written as $(v_1, v_2, \ldots, v_k)$, in which the nodes $v_1, v_2, \ldots, v_k$ are distinct except that possibly $v_1 = v_k$. We use $P^{-1}$ to denote the path $(v_k, v_{k-1}, \ldots, v_2, v_1)$. Let $I(P) = V(P) - \{v_1, v_k\}$ be the set of the internal nodes of $P$. A set of paths $\{P_1, P_2, \ldots, P_q\}$ is internally node-disjoint (abbreviated as disjoint) if $I(P_i) \cap I(P_j) = \emptyset$ for any $i \neq j$. The length of a path $Q$, $l(Q)$, is the number of edges in $Q$. We also write the path $(v_1, v_2, \ldots, v_k)$ as $(v_1, Q_1, v_2, v_3, \ldots, Q_2, v_k)$, where $Q_1$ is the path $(v_1, v_2, \ldots, v_k)$ and $Q_2$ is the path $(v_k, v_{k-1}, \ldots, v_1)$. Hence, it is possible to write a path as $(v_1, Q, v_2, v_3, \ldots, v_k)$ if $l(Q) = 0$. We use $d(u, v)$ to denote the distance between $u$ and $v$, that is, the length of a shortest path joining $u$ and $v$. The diameter of a graph $G$, $D(G)$, is defined as $\max\{d(u, v) | u, v \in V\}$. A path is a hamiltonian path if it contains all nodes of $G$. A graph $G$ is hamiltonian connected if for any two distinct nodes of $G$ there is a hamiltonian path of $G$ between them. A cycle is a path with at least three nodes such that the first node is the same as the last node. A hamiltonian cycle of $G$ is a cycle that traverses every node of $G$ exactly once. A graph is hamiltonian if it has a hamiltonian cycle.

The connectivity of $G$, $\kappa(G)$, is the minimum number of nodes whose removal leaves the remaining graph disconnected or trivial. It follows from Menger’s Theorem [21] that there are $k$ internally node-disjoint paths joining any two distinct nodes $u$ and $v$ when $k \leq \kappa(G)$. A container $C(u, v)$ between two distinct nodes $u$ and $v$ in $G$ is a set of internally disjoint paths $\{P_1, P_2, \ldots, P_r\}$ between $u$ and $v$. The width of $C(u, v)$ is $r$. A $w$-container is a container of width $w$. The length of $C(u, v) = (P_1, \ldots, P_r)\cap C(u, v)$, is $\max\{|l(P_i)| | 1 \leq i \leq r\}$. The $w$-wide distance between $u$ and $v$, $d_w(u, v)$, is $\min\{|l(C(u, v)) | C(u, v)\}$ is a $w$-container. The $w$-diameter of $G$, $D_w(G)$, is $\max\{d_w(u, v) | u, v \in V, u \neq v\}$.

In particular, the wide diameter of $G$ is $D_w(G)$. The wide diameter is used to measure the performance of multipath communication in networks [14].

In this article, we are interested in a specific type of container. We say that a $w$-container $C(u, v)$ is a $w^*$-container if every node of $G$ is incident with a path in $C(u, v)$. A graph $G$ is $w^*$-connected if there exists a $w^*$-container between any two distinct nodes $u$ and $v$. Obviously, a graph $G$ is 1-connected if and only if it is hamiltonian connected. Moreover, a graph $G$ is 2*-connected if it is hamiltonian. The study of $w^*$-connected graphs is motivated by the globally 3*-connected graphs proposed by Albert et al. [2]. A globally 3*-connected graph is a 3-regular 3*-connected graph. Assume that a graph $G$ is 3*-connected. Obviously, $w \leq \kappa(G)$. A graph $G$ is super spanning connected if $G$ is $w^*$-connected for any $w$ with $1 \leq w \leq \kappa(G)$. In [19,26], some families of graphs are proved to be super spanning connected.

Graph containers do exist in engineering designed information and telecommunication networks or in biological and neural systems ([1,14] and its references). The study of $w$-container, $w$-wide distance, and their $w^*$-versions plays a pivotal role in design and implementation of parallel routing and efficient information transmission in large-scale networking systems. In biological informatics and neuroinformatics, the existence and structure of a $w^*$-container signifies the cascade effect in the signal transduction system and the reaction in a metabolic pathway.

A graph $G$ is bipartite if its node set can be partitioned into two subsets $V_1$ and $V_2$ such that every edge joins nodes between $V_1$ and $V_2$. Let $G$ be a bipartite graph with bipartition $V_1$ and $V_2$ such that $|V_1| \geq |V_2|$. Suppose that there exists a $w^*$-container $C(u, v) = (P_1, P_2, \ldots, P_n)$ in $G$ joining $u$ to $v$ with $u, v \in V_1$. Obviously, the number of nodes in $P_i$ is $2t_i + 1$ for some integer $t_i$. There are $t_i - 1$ nodes of $P_i$ in $V_1$ other than $u$ and $v$, and $t_i$ nodes of $P_i$ in $V_2$. As a consequence, $|V_1| = \sum_{i=1}^{n}(t_i - 1) + 2$ and $|V_2| = \sum_{i=1}^{n} t_i$. Therefore, any bipartite graph $G$ with $\kappa(G) \geq 3$ is not $w^*$-connected for any $w$, $3 \leq w \leq \kappa(G)$.

Let $G$ be a $w^*$-laceable bipartite graph with bipartite node sets $V_1$ and $V_2$ and $|V_1 \cup V_2| \geq 2$. From the above discussion, $|V_1| = |V_2|$. For this reason, a bipartite graph is $w^*$-laceable if there exists a $w^*$-container between any two nodes from different partite sets for some $w$, $1 \leq w \leq \kappa(G)$. A 1$^*$-laceable graph is also known as a hamiltonian laceable graph [25]. Moreover, a graph $G$ is $2^*$-laceable if and only if it is hamiltonian. All 1$^*$-laceable graphs except $K_1$ and $K_2$ are $2^*$-laceable. A bipartite graph $G$ is super spanning laceable if $G$ is $\ell^*$-laceable for all $1 \leq \ell \leq \kappa(G)$. Recently, Chang et al. [6] proved that the $n$-dimensional hypercube $Q_n$ is super spanning laceable for every positive integer $n$. It was proved in [19] that the $n$-dimensional star graph $S_n$ is super spanning laceable if and only if $n \neq 3$.

We also define the $w^*$-laceable distance between any two nodes $u$ and $v$ from different partite sets, $d^w_u(v, u)$, as $\min\{|l(C(u, v)) | C(u, v)\}$ is a $w^*$-container. The $w^*$-diameter of $G$, denoted by $D^w_u(G)$, is defined as $\max\{d^w_u(u, v) | u, v \in V, u \neq v\}$.

In particular, the spanning wide diameter of $G$ is $D^w_u(G)$. In this article, we evaluate the spanning wide diameter of $S_n$ and the $2^*$-diameter of $S_n$.

![FIG. 1. The star graphs $S_2$, $S_3$, and $S_4$.](image_url)
In Section 2, we give the definition of star graphs and introduce some basic properties of star graphs. In Section 3, we prove some hamiltonian path properties of star graphs.

Then we discuss the spanning wide diameter of $S_n$, in Section 4. In Section 5, we discuss the $2^n$-spanning diameter of $S_n$.

![Illustration for Lemma 3.](image)

**2. STAR GRAPHS AND THEIR PROPERTIES**

Assume that $n \geq 2$. We use $(n)$ to denote the set \{1, 2, \ldots, n\}, where $n$ is a positive integer. A permutation on $(n)$ is a sequence of $n$ distinct elements $u_i \in (n)$, $u_1 u_2 \ldots u_n$. An inversion of $u_1 u_2 \ldots u_n$ is a pair $(i, j)$ such that $u_i < u_j$ and $i > j$. An even permutation is a permutation with an even number of inversions, and an odd permutation is a permutation with an odd number of inversions. The $n$-dimensional star network, denoted by $S_n$, is a graph with the node set $\{u_1 u_2 \ldots u_n \mid u_i \in (n)\}$ and $u_i \neq u_j$ for $i \neq j$. The edges are specified as follows: $u_1 u_2 \ldots u_i \ldots u_n$ is an edge in dimension $i$ with $2 \leq i \leq n$ if $v_i = u_j$ for $j \neq \{1, i\}$. $v_i = u_1$ and $v_i = v_1$. By definition, $S_n$ is an $(n-1)$-regular graph with $n^2$ nodes. Moreover, it is node transitive and edge transitive [1]. The star graphs $S_2$, $S_3$, and $S_4$ are shown in Figure 1 for an illustration.

We use boldface to denote nodes in $S_n$. Hence, $u_1$, $u_2$, \ldots, $u_n$ denotes a sequence of nodes in $S_n$. We use $e$ to denote the element $12 \ldots n$. It is known that $S_n$ is a bipartite graph with one partite set containing those nodes corresponding to odd permuta-
tions and the other partite set containing those nodes corresponding to even permutations. We will use white nodes to represent those even permutation nodes and use black nodes to represent those odd permutation nodes. Let $u = u_1 u_2 \ldots u_n$ be any node of the star graph $S_n$. We say that $u_i$ is the $i$-th coordinate of $u$, denoted by $u_i$, for $1 \leq i \leq n$. By the definition of $S_n$, there is exactly one neighbor $v$ of $u$ such that $u$ and $v$ are adjacent through an $i$-dimensional edge with $2 \leq i \leq n$. For this reason, we use $(u)$ to denote the unique $i$-neighbor of $u$. Obviously, $(u)^i = u$. For $1 \leq i \leq n$, let $S_n^{[i]}$ denote the subgraph of $S_n$ induced by those nodes $u$ with $(u)_i = i$. Obviously, $S_n$ can be decomposed into $n$ sub-
graphs $S_n^{[i]}$, $1 \leq i \leq n$, and each $S_n^{[i]}$ isomorphic to $S_{n-1}$. Thus, the star graph can be constructed recursively. Obviously, $u \in S_n^{[u_1 \ldots u_i]}$ and $(u)^n \in S_n^{[u_1 \ldots u_n]}$. Let $I \subseteq (n)$. We use $S_n^I$ to denote the subgraph of $S_n$ induced by $\cup_{i \in I} V(S_n^{[i]})$. For $1 \leq i \neq j \leq n$, we use $E_{ij}$ to denote the set of edges between $S_n^{[i]}$ and $S_n^{[j]}$. For $1 \leq i \neq j \leq n$, we use $S_n^{[ij]}$ to denote the subgraph of $S_n$ induced by those nodes $u$ with $(u)_{n-1} = i$ and $(u)_n = j$. Obviously, $S_n^{[ij]} \neq S_n^{[ji]}$ and $S_n^{[ij]}$ is isomorphic to $S_{n-2}$.

**Lemma 1** ([23]). Assume that $n \geq 4$, $|E_{ij}| = (n-2)!$ for any $1 \leq i \neq j \leq n$. Moreover, there are $\frac{(n-2)!}{2}$ edges joining black nodes of $S_n^{[i]}$ to white nodes of $S_n^{[j]}$.

**Lemma 2** ([1]). Let $u$ and $v$ be any two distinct nodes of $S_n$ with $d(u, v) \leq 2$. Then $(u)_1 \neq (v)_1$. Moreover, $\{(u)_1 \mid 2 \leq i \leq n-1\} = (n) - \{(u)_1, (u)_n\}$ if $n \geq 3$.

**3. HAMILTONIAN PATHS OF STAR GRAPHS**

**Theorem 1** ([1]). $S_n$ is hamiltonian laceable if and only if $n \neq 3$.

**Theorem 2** ([118]). Suppose that $w$ is an odd black node of $S_n$ with $n \geq 4$. Then there is a hamiltonian path $P$ of $S_n - \{w\}$ joining any two distinct white nodes $u$ and $v$.

**Lemma 3.** Let $n \geq 5$ and $I = \{a_1, a_2, \ldots, a_r\}$ be a subset of $(n)$ for some $r \in (n)$. Assume that $u$ is a white node in $S_n^{[a_1]}$ and $v$ is a black node in $S_n^{[a_r]}$. Then there is a hamiltonian path $P$ of $S_n$ joining $u$ to $v$.

**Proof.** Let $x_1 = u$ and $y_r = v$. Obviously, $S_n^{[a_k]}$ is isomorphic to $S_{n-1}$ for every $i \in (r)$. By Theorem 1, this result holds on $r = 1$. Suppose that $r \geq 2$. By Lemma 1, there are $(n-2)!/2 \geq 3$ edges joining black nodes of $S_n^{[a_1]}$ to white nodes of $S_n^{[a_2]}$. $\forall i \in (r)$. For every $i \in (r-1)$, we can choose a black node $y_1 \in S_n^{[a_1]}$ and a white node $x_{i+1} \in S_n^{[a_1]}$ such that $(y_1, x_{i+1}) \in E_{a_i a_{i+1}}$. By Theorem 1, there is a hamiltonian path $H_i$ of $S_n^{[a_1]}$ joining $x_i$ to $y_{i+1}$ for any $i \in (r)$. Then $(x_1, H_1, y_1, x_2, H_2, y_2, \ldots, x_r, H_r, y_r)$ forms the desired hamiltonian path of $S_n$ joining $u$ to $v$. See Figure 2 for an illustration.

**Lemma 4.** Assume that $r$ and $s$ are any two adjacent nodes of $S_n$ with $n \geq 4$. Then, for any white node $u$ in $S_n - \{r, s\}$ and for any $i \in (n)$, there exists a hamiltonian path $P$ of $S_n - \{r, s\}$ joining $u$ to some black node $v$ of $S_n - \{r, s\}$ with $(v)_1 = i$.

**Proof.** Because $S_n$ is node transitive and edge transitive, we assume that $r = e$ and $s = (e)^2$. Obviously, both $e$ and $(e)^2$ are in $S_n^{[i]}$. We prove this lemma by induction on $n$. Suppose that $n = 4$. The required hamiltonian paths of $S_4 - \{1234, 2134\}$ are listed below.
Assume that this result holds in $S_i$ for every $4 \leq k \leq n - 1$. We have the following cases:

**Case 1.** $u \in S_k^{[n]}$. By induction, there is a Hamiltonian path $P$ of $S_k^{[n]} - \{e, (e)^2\}$ joining $u$ to a black node $x$ with $(x)_1 = n - 1$. Note that $(x)^h$ is a white node of $S_k^{(n-1)}$. We choose a black node $v$ in $S_n^{(n-2)}$ with $(v)_1 = i$. By Lemma 3, there is a Hamiltonian path $Q$ of $S_n^{(n-2)} - (x)^h$ joining $(x)^h$ to $v$. Then $(u, P, x, (x)^h, Q, v)$ forms the desired Hamiltonian path of $S_k^{[n]} - \{e, (e)^2\}$ joining $u$ to $v$ with $(v)_1 = i$. See Figure 3(a) for an illustration.

**Case 2.** $u \in S_k^{[n]}$ for some $k \in (n - 1)$. By Lemma 1, there are $(n - 2)/2 \geq 3$ edges joining black nodes of $S_k^{(k)}$ to white nodes of $S_k^{[n]}$. We can choose a white node $x \in S_k^{[n]} - \{e, (e)^2\}$ with $(x)_k = k$. By Theorem 1, there is a Hamiltonian path $P$
Theorem 3. Let \( n \geq 5 \) and \( I = \{a_1, a_2, \ldots, a_r\} \) be a subset of \( n \) for some \( r \in \mathbb{N} \). Then \( S_n^I \) is Hamiltonian laceable.

Proof. Let \( u \) be a white node and \( v \) be a black node of \( S_n^I \).

Case 1. \( v^n \in S_n^I \). Without loss of generality, we assume that \( u_n = (v_n) = a_1 \). Let \( a_1 \in S_n^I \). By Lemma 2, there is a Hamiltonian path \( P \) of \( S_n^I \) joining \( u \) to a white node \( x \) with \( (x_1) = a_2 \). By Lemma 3, there is a Hamiltonian path \( Q \) of \( S_n^I \) joining the black node \( x^n \) to the white node \( y^n \). Then \( (u, P, x, y, v) \) is the desired Hamiltonian path of \( S_n^I \) joining \( u \) to \( v \). See Figure 4(a) for an illustration.

Case 2. \( u^n \notin S_n^I \) and \( v^n \notin S_n^I \). We can choose a white node \( y \) with \( y \) being a neighbor of \( v \) in \( S_n^I \). Obviously, \( y \neq u \). By Lemma 4, there is a Hamiltonian path \( P \) of \( S_n^{(1)} \) joining \( u \) to a black node \( x \) with \( (x_1) = a_2 \). By Lemma 3, there is a Hamiltonian path \( Q \) of \( S_n^{(1)} \) joining the white node \( x^n \) to the black node \( y^n \). Then \( (u, P, x, y, v) \) is the desired Hamiltonian path of \( S_n^I \) joining \( u \) to \( v \). See Figure 4(b) for an illustration.

Theorem 4. Assume that \( r \) and \( s \) are adjacent nodes of \( S_n \) with \( n \geq 5 \). Then \( S_n - \{r, s\} \) is Hamiltonian laceable.

Proof. Because \( S_n \) is node transitive and edge transitive, we assume that \( r = e \) and \( s = (e)^2 \). Obviously, both \( e \) and \( (e)^2 \) are in \( S_n \). Let \( u \) be a white node and \( v \) be a black node of \( S_n - \{e, (e)^2\} \). We want to find a Hamiltonian path of \( S_n - \{e, (e)^2\} \) joining \( u \) to \( v \).

Case 1. \( u, v \in S_n \). By Lemma 4, there is a Hamiltonian path \( P \) of \( S_n^{(1)} \) joining \( u \) to a black node \( y \) with \( (y_1) = 1 \). We write \( P \) as \( (u, Q_1, x, v, Q_2, y) \). (Note that \( l(Q_1) = 0 \) if \( u = x \) and \( l(Q_2) = 0 \) if \( v = y \).) By Theorem 3, there is a Hamiltonian path \( R \) of \( S_n^{(n-1)} \) joining the black node \( y^n \) to the white node \( y^n \). Then \( (u, Q_1, x, (x)^n, R, (y)^n, y, (Q_2, y, v) \) is the desired Hamiltonian path of \( S_n - \{e, (e)^2\} \) joining \( u \) to \( v \). See Figure 5(a) for an illustration.

Case 2. \( u, v \in S_n^{(k)} \) for some \( k \in \{n-1\} \). By Theorem 1, there is a Hamiltonian path \( P \) of \( S_n^{(k)} \) joining \( u \) to \( v \). By Lemma 1, there are \( (n-2)/2 \geq 3 \) edges joining white nodes of \( S_n^{(k)} \) to black nodes of \( S_n^{(k)} \). We choose a white node \( x \) of \( S_n^{(k)} \), with \( x^n \) being a black node of \( S_n^{(k)} \). We write \( P \) as \( (u, Q_1, x, y, Q_2, v) \). (Note that \( l(Q_1) = 0 \) if \( u = x \) and \( l(Q_2) = 0 \) if \( v = y \).) By Lemma 2, \( (y_1) \in \{n-1\} \). By Theorem 4, there is a Hamiltonian path \( R \) of \( S_n^{(k)} \) joining \( (x)^n \) to a white node \( z \) with \( (z_1) \in \{n-1\} \). By Theorem 3, there is a Hamiltonian path \( T \) of \( S_n^{(n-1)} \) joining the black node \( (y)^n \) to the white node \( (y)^n \). Then \( (u, Q_1, x, (x)^n, R, z, (z)^n, y, Q_2, v) \) is the desired Hamiltonian path of \( S_n - \{e, (e)^2\} \) joining \( u \) to \( v \). See Figure 5(b) for an illustration.

Case 3. \( u \in S_n^{(k)} \) and \( v \in S_n^{(l)} \) for some \( k \in \{n-1\} \). By Theorem 4, there is a Hamiltonian path \( P \) of \( S_n^{(k)} \) joining \( u \) to a black node \( x \) with \( (x_1) \in \{n-1\} \). By Theorem 3, there is a Hamiltonian path \( Q \) of \( S_n^{(l)} \) joining the black node \( x^n \) to the white node \( y^n \). Then \( (u, P, x, (x)^n, Q, v) \) is the desired Hamiltonian path of \( S_n - \{e, (e)^2\} \) joining \( u \) to \( v \). See Figure 5(c) for an illustration.

Case 4. \( u \in S_n^{(k)} \) and \( v \in S_n^{(l)} \) with \( k, l \), and \( n \) being distinct. By Lemma 1, there are \( (n-2)/2 \geq 3 \) edges joining black nodes of \( S_n^{(k)} \) to white nodes of \( S_n^{(l)} \). We choose a black node \( x \) of \( S_n^{(k)} \) with \( x^n \) being a white node of \( S_n^{(l)} \). By Theorem 1, there is a Hamiltonian path \( P \) of \( S_n^{(k)} \) joining \( u \) to \( x \). By Lemma 4, there is a Hamiltonian path \( Q \) of \( S_n^{(l)} \) joining \( x^n \) to a black node \( y \) with \( (y_1) \in \{n-1\} \). By Theorem 3, there is a Hamiltonian path \( R \) of \( S_n^{(n-1)} \) joining \( x^n \) to a black node \( z \) with \( (z_1) \in \{n-1\} \). By Theorem 3, there is a Hamiltonian path \( T \) of \( S_n^{(n-1)} \) joining \( x^n \) to a black node \( y^n \). Then \( (u, Q_1, x, (x)^n, R, z, (z)^n, y, Q_2, v) \) is the desired Hamiltonian path of \( S_n - \{e, (e)^2\} \) joining \( u \) to \( v \). See Figure 5(d) for an illustration.

FIG. 4. Illustration for Theorem 3.

FIG. 5. Illustration for Theorem 4.
the white node \((y)^n\) to \(v\). Then \((u,P,x,(x)^n,Q,y,(y)^n,R,v)\) is the desired hamiltonian path of \(S_n - \{e, (e)^2\}\) joining \(u\) to \(v\).

See Figure 5(d) for an illustration.

Lemma 5. Assume that \(n \geq 5\). Suppose that \(p\) and \(q\) are two different white nodes of \(S_n\), and \(r\) and \(s\) are two different black nodes of \(S_n\). Then there exist two disjoint paths \(P_1\) and \(P_2\) such that (1) \(P_1\) joins \(p\) to \(r\), (2) \(P_2\) joins \(q\) to \(s\), and (3) \(P_1 \cup P_2\) spans \(S_n\).

Proof. Because \(S_n\) is edge transitive, we assume that \(p \in S_n^{(n)}\) and \(q \in S_n^{(n-1)}\). Suppose that \(r \in S_n^{(n)}\) and \(s \in S_n^{(n)}\).

Case 1. \(i,j \in \langle n - 2\rangle\) with \(i \neq j\). By Theorem 3, there is a hamiltonian path \(P_1\) of \(S_n^{[m]}\) joining \(p\) to \(r\). Again, there is a hamiltonian path \(P_2\) of \(S_n^{[m-1]}\) joining \(q\) to \(s\). Then \(P_1\) and \(P_2\) are the desired paths.

Case 2. \(i,j \in \langle n - 2\rangle\) with \(i = j\). We can choose a white node \(x\) with \(x\) being a neighbor of \(s\) in \(S_n^{(i)}\) and \((x)_1 \in \langle n - 1\rangle - \{i\}\). By Lemma 4, there is a hamiltonian path \(Q\) of \(S_n^{[m]} - \{s,x\}\) joining \(r\) to a white node \(y\) with \((y)_1 = n\). By Theorem 3, there is a hamiltonian path \(P\) of \(S_n^{[m]}\) joining \(p\) to the black node \((x)^n\). Moreover, there is a hamiltonian path \(R\) of \(S_n^{[m-1]}\) joining \(q\) to the black node \((x)^n\). Then \(P_1 = (p,P,(x)^n,y,1,R)\) and \(P_2 = (q,R,(y)^n,x,s)\) are the desired paths.

Case 3. Either \((i = n\) and \(j \in \langle n - 1\rangle\)) or \((i \in \langle n\rangle - \{n - 1\}\) and \(j = n - 1\)). By symmetry, we assume that \(i = n\) and \(j \in \langle n - 1\rangle\). By Theorem 3, there is a hamiltonian path \(P_1\) of \(S_n^{[n]}\) joining \(p\) to \(r\). Moreover, there is a hamiltonian path \(P_2\) of \(S_n^{[n-1]}\) joining \(q\) to \(s\). Then \(P_1\) and \(P_2\) are the desired paths.

Case 4. Either \((i = n - 1\) and \(j \in \langle n - 2\rangle\)) or \((i \in \langle n - 2\rangle\) and \(j = n\)). By symmetry, we assume that \(i = n - 1\) and \(j \in \langle n - 2\rangle\). By Lemma 1, there exist \((n - 2)!/2 \geq 3\) edges joining white nodes of \(S_n^{(n-1)}\) to black nodes of \(S_n^{(n)}\). We can choose a white node \(x\) in \(S_n^{(n-1)}\) with \((x)_1 = n\). By Theorem 3, there is a hamiltonian path \(R\) of \(S_n^{(n-1)}\) joining \(q\) to \(r\). We write \(R\) as \((q,R_1,y,x,R_2,r)\). By Theorem 3, there is a hamiltonian path \(P\) of \(S_n^{[n]}\) joining \(p\) and the black node \((x)^n\). Because \(d(x,y) = 1\), by Lemma 2, \((y)^n \in S_n^{(n-2)}\). By Theorem 3, there exists a hamiltonian path \(Q\) of \(S_n^{(n-2)}\) joining the white node \((y)^n\) to \(s\). Then \(P_1 = (p,P,(x)^n,x,R_2,r)\) and \(P_2 = (q,R_1,y,(y)^n,1)\) are the desired paths.

Case 5. \(i = n - 1\) and \(j = n\). By Theorem 3, there is a hamiltonian path \(Q\) of \(S_n^{[n]}\) joining \(p\) to \(s\). Again, there is a hamiltonian path \(R\) of \(S_n^{[n-1]}\) joining \(q\) to \(r\). We choose a white node \(x\) in \(S_n^{[n]}\) with \((x)_1 = n - 1\). We write \(Q\) as \((q,R_1,w,(x)^n,2R)\). Obviously, \(y\) is a black node and \(w\) is a white node. Because \(d(x,y) = 1\), by Lemma 2, \((y)_1 = \langle n - 2\rangle\). Because \(d((x)^n,w) = 1\), by Lemma 2, \((w)_1 \in \langle n - 2\rangle\). By Theorem 3, there exists a hamiltonian path \(W\) of \(S_n^{[n-1]}\) joining the black node \((w)^n\) to the white node \((y)^n\). Then \(P_1 = (p,R_1,x,(x)^n,R_2,r)\) and \(P_2 = (q,R_1,w,(w)^n,W,y,1,R_2,s)\) are the desired paths.

Case 6. Either \(i = j = n\) or \(i = j = n - 1\). By symmetry, we assume that \(i = j = n\). By Theorem 3, there is a hamiltonian path \(P\) of \(S_n^{[n]}\) joining \(p\) to \(s\). We can write \(P\) as \((p,R_1,r,x,R_2,s)\). By Theorem 3, there is a hamiltonian path \(Q\) of \(S_n^{[n-1]}\) joining \(q\) to the black node \((y)^n\). Then \(P_1 = (p,R_1,r)\) and \(P_2 = (q,Q,(x)^n,x,R_2,s)\) are the desired paths.

4. THE \((n - 1)^{\text{st}}\)-DIAMETER OF \(S_n\)

Let \(u\) be a node of \(S_n\) with \(n \geq 4\) and let \(m\) be any integer with \(3 \leq m \leq n\). We set \(F_m(u) = \{(u)^i | 3 \leq i \leq m\} \cup \{((u)^i)^{\text{th}} \mid 3 \leq i \leq m\}\).

Lemma 6. Assume that \(u\) is a white node of \(S_n\) and \(j \in (n)\) with \(n \geq 4\). Then there is a hamiltonian path \(P\) of \(S_n - F_m(u)\) joining \(u\) to some black node \(v\) with \((v)_1 = j\).

Proof. We prove this lemma by induction on \(n\). Because \(S_n\) is node transitive, we assume that \(u = e\). Suppose that \(n = 4\). The required hamiltonian paths of \(S_4 - F_4(e)\) are listed below:

Assume that this statement holds on any \(S_k\) for every \(4 \leq k \leq n - 1\). We have \(F_m(e) = F_{m-1}(e) \cup \{(e)^n,((e)^n)^{n-1}\}\). By induction, there is a hamiltonian path \(P\) of \(S_n^{(n)} - F_{m-1}(e)\) joining \(e\) to a black node \(x\) with \((x)_1 = 1\). By Lemma 4, there is a hamiltonian path \(Q\) of \(S_n^{[n-1]} - \{(e)^n,((e)^n)^{n-1}\}\) joining the white node \((x)^n\) to a black node \(y\) with \((y)_1 = 2\). We can choose a black node \(z\) of \(S_n^{(n-1)}\) with \((z)_1 = j\). By Theorem 3, there exists a hamiltonian path \(R\) of \(S_n^{(n-1)}\) joining the white node \((y)^n\) to \(z\).

Then \((e,P,x,(x)^n,Q,y,(y)^n,R,z)\) is a desired hamiltonian path.

Lemma 7. Let \(u = u_1u_2u_3u_4\) be any white node of \(S_4\). There exist three paths \(P_1, P_2,\) and \(P_3\) such that (1) \(P_1\) joins \(u\) to the black node \(u_2u_3u_4\) with \(l(P_1) = 7\), (2) \(P_2\) joins \(u\) to the white node \(u_1u_3u_4\) with \(l(P_2) = 8\), (3) \(P_3\) joins \(u\) to the white node \(u_1u_2u_4\) with \(l(P_3) = 8\), and (4) \(P_1 \cup P_2 \cup P_3\) spans \(S_4\).
Proof. Because $S_4$ is node transitive, we assume that $u = 1234$. Then we set

$$P_1 = (1234, 3214, 4213, 1243, 2143, 4123, 1423, 2413),$$
$$P_2 = (1234, 4231, 3241, 2341, 4321, 3421, 2431, 1432, 3412),$$
$$P_3 = (1234, 2134, 3124, 1324, 2314, 4312, 1342, 3142, 4132).$$

Obviously, $P_1$, $P_2$, and $P_3$ are the desired paths.

Lemma 8. Let $u = u_1 u_2 u_3 u_4$ be any white node of $S_4$. Let $i_1 i_2 i_3 i_4$ be a permutation of $u_2$, $u_3$, and $u_4$. There exist four paths $P_1$, $P_2$, $P_3$, and $P_4$ of $S_4$ such that (1) $P_1$ joins $u$ to a white node $w$ with $(w_1) = i_1$ and $l(P_1) = 2$, (2) $P_2$ joins $u$ to a white node $x$ with $(x_1) = i_2$ and $l(P_2) = 2$, (3) $P_3$ joins $u$ to a black node $y$ with $(y_1) = i_3$ and $l(P_3) = 19$, (4) $P_4$ joins $u$ to a black node $z$ with $z \neq y$, $(z_1) = i_4$, and $l(P_4) = 19$.

Proof. Because $S_4$ is node transitive, we assume that $u = 1234$. Because $u = 1234$, we have $(i_1, i_2) \subset \{2, 3, 4\}$ and $i_3 \in \{2, 3, 4\} - (i_1, i_2)$. Without loss of generality, we suppose that $i_1 < i_2$. The required four paths are listed below.

<table>
<thead>
<tr>
<th>$i_1$</th>
<th>$i_2$</th>
<th>$i_3$</th>
<th>$i_4$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>$(1234, 4231, 2341)$</td>
<td>$(1234, 3241, 2431)$</td>
<td>$(1234, 2134, 3412)$</td>
<td>$(1234, 2431, 1342)$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>$(1234, 4321, 3241)$</td>
<td>$(1234, 3421, 2341)$</td>
<td>$(1234, 2314, 4321)$</td>
<td>$(1234, 2143, 3421)$</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>$(1234, 4231, 3421)$</td>
<td>$(1234, 3241, 2431)$</td>
<td>$(1234, 2134, 3412)$</td>
<td>$(1234, 2431, 1342)$</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>$(1234, 4231, 3241)$</td>
<td>$(1234, 3421, 2341)$</td>
<td>$(1234, 2314, 4321)$</td>
<td>$(1234, 2143, 3421)$</td>
</tr>
</tbody>
</table>

Thus, this statement is proved.

Lemma 9. Assume that $n \geq 5$ and $i_1 i_2 \cdots i_{n-1}$ is an $(n-1)$-permutation on $(n)$. Let $u$ be any white node of $S_n$. Then there exist $(n-1)$ paths $P_1, P_2, \ldots, P_{n-1}$ of $S_n$ such that (1) $P_1$ joins $u$ to a black node $y_1$ with $(y_1)_1 = i_1$ and $l(P_1) = n(n-2)! - 1$, (2) $P_2$ joins $u$ to a white node $y_2$ with $(y_2)_1 = i_2$ and $l(P_2) = n(n-2)!$ for every $2 \leq j \leq n - 1$, (3) $\cup_{j=1}^{n-1} P_j$ spans $S_n$, and (4) $\cap_{j=1}^{n-1} V(P_j)$ $=$ $\{u\}$.

Proof. The proof of this lemma is rather tedious. The authors strongly suggest the reader skim over the proof first and comprehend the details later.

Because $S_n$ is node transitive, we assume that $u = e$. Without loss of generality, we suppose that $i_2 < i_3 < \ldots < i_{n-1}$.

Case 1. $n = 5$. Hence, $n(n-2)! = 30$. We have $i_2 \neq 4$, $i_3 \geq 2$, and $i_4 \geq 3$. We set $x_1 = (e)^5$ and $x_i = ((x_{i-1})^i)^5$ for every $2 \leq i \leq 4$, and $x_5 = ((x_4)^5)^5$. Note that $x_i$ is a black node in $S_{n-1}$ for every $i \in \{4\}$ and $x_5$ is a black node in $S_{n-1}$. Obviously, $x_1 \neq x_5$. We set $H = \{e, x_1, (x_1)^2, x_2, (x_2)^3, x_3, (x_3)^4, x_4, (x_4)^5, x_5\}$.

Case 1.1. $i_1 = 3$. We have $i_2 \neq 4$, $i_3 \neq 4$, and $i_4 \neq 1$. Let $u_1 = 24135$, $u_2 = 41325$, and $u_3 = 34125$. We set

$$W_1 = \{e = 12345, 32415, 42135, 24135, 21435, 14235, 42135, 24135 = u_1\},$$
$$W_2 = \{e = 12345, 21435, 32415, 12435, 23145, 34215, 13245, 41235 = u_2\},$$
$$W_3 = \{e = 12345, 32415, 42135, 24135, 21435, 13425, 34125, 41325 = u_3\}.$$

Obviously, $W_1 \cup W_2 \cup W_3$ spans $S_5^{[4]}$ and $V(W_1) \cap V(W_2) = \{e\}$ for every $i, j \in \{3\}$ with $i \neq j$. By Lemma 4, there exists a hamiltonian path $Q_1$ of $S_5^{[4]} - (x_2, (x_2)^3)$ joining the white node $(u_1)^5$ to a black node $y_1$ with $(y_1)_1 = i_1$. Again, there exists a hamiltonian path $Q_2$ of $S_5^{[4]} - (x_4, (x_4)^5)$ joining the black node $(u_2)^5$ to a white node $y_2$ with $(y_2)_1 = i_2$. Moreover, there exists a hamiltonian path $Q_3$ of $S_5^{[4]} - (x_3, (x_3)^4)$ joining the black node $(u_3)^5$ to a white node $y_3$ with $(y_3)_1 = i_3$. Similarly, there exists a hamiltonian path $Q_4$ of $S_5^{[4]} - (x_1, (x_1)^5)$ joining the black node $x_5$ to a white node $y_4$ with $(y_4)_1 = i_4$. We set

$$P_1 = \{e, W_1, u_1, (u_1)^5, Q_1, y_1\},$$
$$P_2 = \{e, W_2, u_2, (u_2)^5, Q_2, y_2\},$$
$$P_3 = \{e, W_3, u_3, (u_3)^5, Q_3, y_3\},$$
$$P_4 = \{e, H, x_5, Q_4, y_4\}.$$
Obviously, \( l(P_1) = 29 \) and \( l(P_2) = 30 \) for every \( 2 \leq i \leq 4 \). Apparently, \( P_1, P_2, P_3, \) and \( P_4 \) are the desired paths. See Figure 6(a) for an illustration.

CASE 1.2. \( i_1 \neq 3 \). We have \( i_2 \neq 4, i_3 \neq 1, \) and \( i_4 \neq 2 \). Let \( u_1 = 31425, u_2 = 42135, \) and \( u_3 = 21435 \). We set

\[
W_1 = (e = 12345, 21345, 41325, 14325, 34125, 43125, 13425, 31425 = u_1),
\]
\[
W_2 = (e = 12345, 32145, 23145, 13245, 31245, 41235, 41235 = u_2),
\]
\[
W_3 = (e = 12345, 42315, 24315, 34215, 43215, 32415, 12435 = u_3).
\]

Obviously, \( W_1 \cup W_2 \cup W_3 \) spans \( S_{5}^{(0)} \) and \( V(W_i) \cap V(W_j) = \{ e \} \) for every \( i, j \) with \( i \neq j \). By Lemma 4, there exists a hamiltonian path \( Q_1 \) of \( S_{5}^{(3)} \) joining the white node \( (u_1)^5 \) to a black node \( y_1 \) with \( (y_1)^1 = i_1 \). Again, there exists a hamiltonian path \( Q_2 \) of \( S_{5}^{(4)} \) joining the black node \( (u_2)^5 \) to a white node \( y_2 \) with \( (y_2)^1 = i_2 \). Moreover, there exists a hamiltonian path \( Q_3 \) of \( S_{5}^{(1)} \) joining the black node \( x_3 \) to a white node \( y_3 \) with \( (y_3)^1 = i_3 \). Similarly, there exists a hamiltonian path \( Q_4 \) of \( S_{5}^{(2)} \) joining the black node \( (u_3)^5 \) to a white node \( y_4 \) with \( (y_4)^1 = i_4 \). We set

\[
P_1 = (e, W_1, u_1, (u_1)^5, Q_1, y_1),
\]
\[
P_2 = (e, W_2, u_2, (u_2)^5, Q_2, y_2),
\]
\[
P_3 = (e, H, x_3, Q_3, y_3), \quad \text{and}
\]
\[
P_4 = (e, W_3, u_3, (u_3)^5, Q_4, y_4).
\]

Obviously, \( l(P_1) = 29 \) and \( l(P_2) = 30 \) for every \( 2 \leq i \leq 4 \). Apparently, \( P_1, P_2, P_3, \) and \( P_4 \) are the desired paths. See Figure 6(b) for an illustration.

CASE 2. \( n \geq 6 \). Because \( n - 1 \geq 5 \), we have \( i_k \neq k + 2 \) for every \( 2 \leq k \leq n - 4 \), \( i_{n-3} \neq 1 \), \( i_{n-2} \neq 2 \), and \( i_n \neq 3 \). We set \( u_j = (e)^{j+2} \) and \( v_j = (e)^{j+2}j+1 \) for every \( j \in (n - 4) \). Thus, \( u_j \) is a black node in \( S_{n}^{(0)} \) and \( v_j \) is a white node in \( S_{n}^{(n-1)} \) for every \( j \in (n - 4) \). Note that \( P_{n-2} = (e, u_j, y_j | j \in (n - 4) \cup (v_j | j \in (n - 4)) \). By Lemma 6, there is a hamiltonian path \( P_n \) of \( S_{n}^{(n-1)} \) joining the black node \( (x_1)^n \) to a black node \( x_1 \) with \( (x_1)^1 = 2 \). We recursively set \( x_1 \) as the unique neighbor of \( (x_1)^1 \) in \( S_{n}^{(n-1)} \) with \( (x_1)^1 = j+1 \) for every \( 2 \leq j \leq n - 4 \), and we set \( x_{n-3} \) as the unique neighbor of \( (x_1)^{n-1} \) in \( S_{n}^{(n-3)} \) with \( (x_1)^{n-3} = n - 1 \). It is easy to see that \( x_j \) is a black node for \( 1 \leq j \leq n - 3 \) and \( \{ (x_j)^n, (x_j)^{n+1} \} \subset S_{n}^{(n-1)} \) for \( 1 \leq j \leq n - 4 \). We construct \( P_n \) for every \( 1 \leq j \leq n - 1 \) as follows:

1. \( j \in (n - 4) - \{ 1 \} \). By Lemma 4, there is a hamiltonian path \( T_1 \) of \( S_{n}^{(n-1)} \) joining the black node \( (x_1)^n \) to a white node \( x_1 \) with \( (x_1)^1 = j + 2 \). Again, there is a hamiltonian path \( T_2 \) of \( S_{n}^{(n-1)} \) joining the black node \( (y_1)^n \) to a white node \( y_1 \) with \( (y_1)^1 = i_1 \). Then we set \( P_j = (e, u_j, v_j, (y_1)^{n-1} \cup T_1 \cup T_2, (y_1)^n) \). Obviously, \( l(P_j) = n(n - 2) \).

2. \( j = n - 3 \). We choose a white node \( y_{n-3} \) in \( S_{n}^{(0)} \) with \( (y_{n-3}^1) = i_1 \). Note that there are \( \{ (n - 3) / 2 \} \) edges joining some black nodes of \( S_{n}^{(n-2)} \) to some white nodes of \( S_{n}^{(0)} \) and there are \( \{ (n - 3) / 2 \} \) edges joining some white nodes of \( S_{n}^{(n-2)} \) to some black nodes of \( S_{n}^{(0)} \). We choose a white node \( r \) in \( S_{n}^{(0)} \) with \( (r)^n \) being a black node in \( S_{n}^{(n-2)} \) and choose a black node \( s \) in \( S_{n}^{(0)} \) with \( (s)^n \) being a white node in \( S_{n}^{(n-2)} \).

3. \( j = n - 1 \). By Lemma 4, there is a hamiltonian path \( Q_1 \) of \( S_{n}^{(n-1)} \) joining the black node \( (x_1)^n \) to a white node \( q \) with \( (q)^1 = 3 \). Again, there is a hamiltonian path \( Q_3 \) of \( S_{n}^{(n-1)} \) joining the black node \( (y_1)^n \) to a white node \( y_{n-1} \) with \( (y_{n-1})^1 = i_1 \). We set \( P_n = (e, u_1, v_1, (y_1)^{n-1} \cup Q_1 \cup Q_3, (q)^n, Q_2, y_{n-1}) \). Obviously, \( l(P_n) = n(n - 2) \).

4. We construct \( P_1 \) and \( P_n \) by letting \( n - 1 \) or not. We set \( L \times (x_1)^{n-1}, (x_2)^{n-1}, \ldots, (x_{n-4})^{n-1}, (x_{n-3})^n, (x_{n-2})^n \). By Theorem 1, there is a hamiltonian path \( W \) of \( S_{n}^{(n-1)} \) joining the black node \( (e)^{n-1} \) to a white node \( p \) with \( (p)^1 = 2 \).

Suppose that \( i_{n-1} \neq 1 \). By Theorem 1, there is a hamiltonian path \( R \) of \( S_{n}^{(n-1)} \) joining the white node \( (x_{n-3})^n \) to a black node \( y_1 \) with \( (y_1)^1 = i_1 \). Again, there exists a hamiltonian path \( Z \) of \( S_{n}^{(n-1)} \) joining the black node \( (x_{n-3})^n \) to a white node \( y_{n-2} \) with \( (y_{n-2})^1 = i_{n-2} \). We set \( P_1 = (e, P_1, x_1, L, (x_{n-3})^n, R, y_1) \) and \( P_n = (e, (e)^{n-1}, W, p, (p)^n, Z, y_{n-2}) \). Obviously, \( l(P_1) = n(n - 2)! - 1 \) and \( l(P_{n-2}) = n(n-2)! \). Apparently, \( P_1, P_2, \ldots, P_{n-1} \) are the desired paths. See Figure 7(a) for an illustration for the case \( n = 7 \).
Suppose that $i_1 = n - 1$. Note that $i_{n-2} \neq n - 1$. Because $(x_{n-3})^p$ is a white node in $S_n^{(p-1)}$ with $((x_{n-3})^p)_1 = n$ and $((x_{n-3})^p)_n = n - 1$, there is a black node $z$ in $S_n^{(n-1)}$ such that $z$ is the unique neighbor of $(x_{n-3})^p$ with $(z)_1 = 2$. Because $((x_{n-3})^p)_n = n - 3$, we have $(z)_{n-1} = n - 3$. Note that $(z)^n$ is a white node in $S_n^{(2)}$ with $((z)^n)_{n-1} = n - 3$. Because $(p)^n$ is a black node in $S_n^{(2)}$ with $((p)^n)_1 = n$ and $((p)^n)_n = 2$, there is a white node $t$ in $S_n^{(2)}$ such that $t$ is the unique neighbor of $(p)^n$ with $(t)_1 = n - 1$. Because $(p)_{n-1} = 1$ and $(p)_n = n$, we have $((p)^n)_{n-1} = 1$ and $((p)^n)_1 = n$. Because $((p)^n)_{n-1} = 1$, we have $(t)_{n-1} = 1$. Because $((z)^n)_{n-1} = n - 3$ and $(t)_{n-1} = 1$, we have $(z)^n \neq t$. By Theorem 4, there is a hamiltonian path $W_1$ of $S_n^{(2)} - ((p)^n, t)$ joining $(z)^n$ to a black node $y_1^n$ with $(y_1)_1 = i_1$. Again, there is a hamiltonian path $W_2$ of $S_n^{(n-1)} - ((x_{n-3})^p, z)$ joining $(t)^n$ to a white node $y_{n-2}$ with $(y_{n-2})_1 = i_{n-2}$. We set $P_1$ as $(e, P, x_1, L, (x_{n-3})^n, z, (z)^n, W_1, y_1)$ and $P_{n-2}$ as $(e, (e)^{n-1}, W, (p, (p)^n, t), (t)^n, W_2, y_{n-2})$. Obviously, $l(P_1) = n(n-2)! - 1$ and $l(P_{n-2}) = n(n-2)!$. Apparently, $P_1, P_2, \ldots, P_{n-1}$ are the desired paths. See Figure 7(b) for an illustration for the case $n = 7$.

![FIG. 7. Illustration of case 3 with $n = 7$.](image_url)
Using depth first search, we list all hamiltonian cycles in $S_4$ in Table 1.

**Lemma 10.** $D_5^S(S_4) = 15$.

**Proof.** Let $u$ be any white node in $S_4$, and let $v$ be any black node in $S_4$. Because $S_4$ is node transitive, we assume that $u = 1234$. Suppose that $d(u, v) = 1$. Because $S_4$ is edge transitive, we assume that $v = 2134$. Let $(P_1, P_2, P_3)$ be a 3*-container joining $u$ to $v$. Because $S_4$ is 3-regular, one of three paths, say $P_3$, is $(u, v)$. Thus, $P_1 \cup P_2^{-1}$ forms a hamiltonian cycle of $S_4$ not using the edge $(u, v)$. From Table 1, we obtain $d_5^S(u, v) = 15$. Thus, $D_5^S(S_4) \geq 15$. Suppose that $d(u, v) \neq 1$. Then $v \in \{1234, 1243, 2413, 2431, 3412, 4213, 4312\}$. We find the following set of 3*-containers of $S_4$ between $u = 1234$ and $v$:

| $C(1234, 1324)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |
| $C(1234, 1432)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |
| $C(1234, 2341)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |
| $C(1234, 2341)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |
| $C(1234, 3124)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |
| $C(1234, 3412)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |
| $C(1234, 3421)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |
| $C(1234, 3421)$ | $P_1 = (1234, 4231, 3241, 1243, 2143, 4123, 3124, 1324)$ | $P_2 = (1234, 2134, 4132, 3142, 1342, 2341, 1234)$ | $P_3 = (1234, 3214, 4213, 2413, 1423, 3421, 2341, 1234)$ |

From this table, $d^S_5(u, v) \leq 15$ if $d(u, v) \neq 1$. Hence, $D^S_5(S_4) = 15$.

**Lemma 11.** $D_{n-1}^S(S_4) \geq \frac{n!}{n-2} + 1 = (n-1)! + 2(n-2)! + 2(n-3)! + 1$ if $n \geq 5$.

**Proof.** Let $u$ and $v$ be two adjacent nodes of $S_n$. Obviously, $u$ and $v$ are in different partite sets. Let $(P_1, P_2, \ldots, P_{n-1})$ be any $(n-1)^*$-container of $S_n$ joining $u$ to $v$. Obviously, one of these paths is $(u, v)$. Thus, max $l(P_i) \mid 1 \leq i \leq n-1 \geq \frac{\frac{n!}{n-2}}{2(n-2)!} + 1 = \frac{n!}{n-2} - \frac{1}{2} + 1 = \frac{n!}{n-2} + 1$. Hence, $d^S_{n-1}(u, v) \geq \frac{n!}{n-2} + 1$ and $D_{n-1}^S(S_n) \geq \frac{n!}{n-2} + 1$.

**Lemma 12.** $D^S_4(S_5) \leq 41$.

**Proof.** Let $u$ be any white node and $v$ be any black node of $S_5$. Obviously, $d(u, v)$ is odd.

**CASE 1.** $d(u, v) = 1$. Because $S_5$ is node transitive and edge transitive, we may assume that $u = e = 12345$ and $v = (e) = 52341$. By Lemma 7, there exist three paths $P_1, P_2, P_3$ and $P_5$ of $S_5^S$ such that (1) $P_1$ joins $12345$ to the black node $24135$ with $l(P_1) = 7$, (2) $P_2$ joins $12345$ to the white node $34125$ with $l(P_2) = 8$, (3) $P_3$ joins $12345$ to the white node $34125$ with $l(P_3) = 8$, and (4) $P_1 \cup P_2$ spans $S_5^S$. Similarly, there exist three paths $Q_1, Q_2,$ and $Q_3$ of $S_5^S$ such that (1) $Q_1$ joins $52341$ to the white node $24531$ with $l(Q_1) = 7$, (2) $Q_2$ joins $52341$ to the black node $34521$ with $l(Q_2) = 8$, (3) $Q_3$ joins $52341$ to the black node $45321$ with $l(Q_3) = 8$, and (4) $Q_1 \cup Q_2 \cup Q_3$ spans $S_5^S$. By Theorem 1, there is a hamiltonian path $R_1$ of $S_5^S$. 244 NETWORKS—2006—DOI 10.1002/net
joining the white node 54132 to the black node 14532, there is a Hamiltonian path $R_2$ of $S_5^{[3]}$ joining the black node 54132 to the white node 14532, and there is a Hamiltonian path $R_3$ of $S_5^{[4]}$ joining the black node 51324 to the white node 15324. Then we set

$$T_1 = (e = 12345, P_1, 24135, 54132, R_1, 14532, 24531),$$

$$(Q_1)^{-1}, 52341 = (e)^5,$$

$$T_2 = (e = 12345, P_2, 34125, 54132, R_2, 14523, 34521),$$

$$(Q_2)^{-1}, 52341 = (e)^5,$$

$$T_3 = (e = 12345, P_3, 41325, 51324, R_3, 15324, 45321),$$

$$(Q_3)^{-1}, 52341 = (e)^5,$$

and

$$T_4 = (e = 12345, 52341 = (e)^5).$$

Obviously, $\{T_1, T_2, T_3, T_4\}$ is a $4^*$-container of $S_5$ between $u$ and $v$. Moreover, $l(H_1) = 25$, $l(H_2) = l(H_3) = 29$, and $l(H_4) = 39$. Thus, $d_{4^*}(u, v) \leq 41$.

**SUBCASE 2.2.** $v_1 = e \in \{2, 3\}$. We have $\{(v_2), (v_3), (v_4)\} = \{1, 2, 3, 5\} - \{a\}$. Let $b$ be the only element in $\{2, 3\} - \{a\}$. By Lemma 8, there exist four paths $P_1, P_2, P_3,$ and $P_4$ of $S_5^{[5]}$ such that $P_1$ joins $u$ to a white node $w$ with $(w_1) = 2$ and $l(P_1) = 2$, $P_2$ joins $u$ to a white node $w$ with $(w_1) = 3$ and $l(P_1) = 2$, $P_3$ joins $u$ to a black node $y$ with $(y_1) = 4$ and $l(P_3) = 19$, $(4)$ $P_4$ joins $u$ to a black node $z$ with $(z_1) = 4$ and $l(P_4) = 19$, $(5)$ $P_1 \cup P_2 \cup P_3$ spans $S_5^{[5]}$, $(6)$ $P_1 \cup P_2 \cup P_4$ spans $S_5^{[5]}$, $(7)$ $V(P_1) \cap V(P_2) \cap V(P_3) = \{u\},$ and $(8)$ $V(P_1) \cap V(P_2) \cap V(P_4) = \{u\}$.

Similarly, there exist four paths $Q_1, Q_2, Q_3,$ and $Q_4$ of $S_5^{[5]}$ such that $(Q_1)$ joins $v$ to a black node $w$ with $(w_1) = 2$ and $l(Q_1) = 2$, $(2)$ $Q_2$ joins $v$ to a black node $q$ with $(q_1) = 3$ and $l(Q_1) = 2$, $(3)$ $Q_3$ joins $v$ to a white node $w$ with $(w_1) = 4$ and $l(Q_3) = 19$, $(4)$ $Q_4$ joins $v$ to a white node $w$ with $(w_1) = 4$ and $l(Q_4) = 19$, $(5)$ $Q_1 \cup Q_2 \cup Q_3$ spans $S_5^{[5]}$, $(6)$ $Q_1 \cup Q_2 \cup Q_4$ spans $S_5^{[5]}$, $(7)$ $V(Q_1) \cap V(Q_2) \cap V(Q_3) = \{v\},$ and $(8)$ $V(Q_1) \cap V(Q_2) \cap V(Q_4) = \{v\}$.

By Lemma 11, there are exactly three edges joining some black nodes of $S_5^{[5]}$ to some white nodes of $S_5^{[4]}$. By the pigeon-hole principle, at least one node in $[y, z]$ is adjacent to a node in $[r, s]$. Without loss of generality, we assume that $y$ is adjacent to $r$. Let $T_1$ be the Hamiltonian path of $S_5^{[4]}$ joining the black node $(u)^5$ to the white node $(v)^5$, $T_2$ be the Hamiltonian path of $S_5^{[4]}$ joining the black node $(w)^5$ to the white node $(v)^5$, and $T_3$ be the Hamiltonian path of $S_5^{[4]}$ joining the black node $(q)^5$ to the white node $(q)^5$. We set

$$H_1 = (u, (u)^5, T_1, (v)^5, v),$$

$$H_2 = (u, P_1, w, (w)^5, T_2, (p)^5, p, Q_1^{-1}, v),$$

$$H_3 = (u, P_2, x, (x)^5, T_3, (q)^5, q, Q_2^{-1}, v),$$

and

$$H_4 = (u, P_3, y, r, Q_3^{-1}, v).$$

Obviously, $\{H_1, H_2, H_3, H_4\}$ is a $4^*$-container of $S_5$ between $u$ and $v$. Moreover, $l(H_1) = 25$, $l(H_2) = l(H_3) = 25$, and $l(H_4) = 39$. Thus, $d_{4^*}(u, v) \leq 41$.

**Lemma 13.** $d_{n-1}(u, v) \leq (n - 1)! + (n - 2)! + 2(n - 3)! + 1 = n! \iff 4 \text{ for every } n \geq 6.$

**Proof.** Let $u$ be any white node and $v$ be any black node of $S_n$. Obviously, $d_{4^*}(u, v)$ is odd.

**CASE 2.** $d(u, v) = 1$. Because the star graph is node transitive and edge transitive, we may assume that $u = e$ and $v = (e)^5$. 

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By Lemma 9, there exist \((n-2)\) paths \(P_1, P_2, \ldots, P_{n-2}\) of \(S_1^n\) such that (1) \(P_1\) joins \(e\) to a black node \(x_1\) with \((x_1)_{1} = 2\) and \(l(P_1) = (n-1)-(n-3)-1\), (2) \(P_2\) joins \(e\) to a white node \(x_2\) with \((x_2)_{1} = 1 + 1\) and \(l(P_2) = (n-1)-(n-3)\) for \(2 \leq i \leq n-2\), (3) \(\cup_{i=1}^{n-2}P_i\) spans \(S_1^n\), and (4) \(\cap_{i=1}^{n-2}V(P_i) = \{e\}\). Again, there exist \((n-2)\) paths \(Q_1, Q_2, \ldots, Q_{n-2}\) of \(S_1^n\) such that (1) \(Q_1\) joins \(e\) to a black node \(y_1\) with \((y_1)_{1} = 2\) and \(l(Q_1) = (n-1)-(n-3)-1\), (2) \(Q_2\) joins \(e\) to a black node \(y_2\) with \((y_2)_{1} = 1 + 1\) and \(l(Q_2) = (n-1)-(n-3)\) for \(2 \leq i \leq n-2\), (3) \(\cup_{i=1}^{n-2}Q_i\) spans \(S_1^n\), and (4) \(\cap_{i=1}^{n-2}V(Q_i) = \{y\}\).

By Theorem 1, there is a Hamiltonian path \(R_1\) of \(S_1^n\) joining the white node \((x)_{n-1}\) to the black node \((y)_{n-1}\). Again, there is a Hamiltonian path \(R_1\) joining the black node \((x)_{n-1}\) to the black node \((y)_{n-1}\) for every \(2 \leq i \leq n-2\).

We set \(H_i = (e, P_i, x_1, (x_i)_{n}, R_i, (y_i)_{n}, y_1, Q_i, (e)_{n})\) for every \(1 \leq i \leq n-2\) and \(H_{n-1} = (e, (e)_{n})\). Then \(H_1, H_2, \ldots, H_{n-1}\) is an \((n-1)\)-container between \(e\) and \((e)_{n}\). Obviously, \(l(H_i) = (n-1)\) for \(2 \leq i \leq n-2\), and \(l(H_{n-1}) = 1\). Hence, \(d_{n-1}^{-}\) is a \((n-1)\)-container between \((e)_{n-1}\) and \((e)_{n}\). Obviously, \(T_1 = \{(u, P_1, x_1, (x_1)_{n}, H_1, (y)_{n}, y_1, Q_1^{-1}, v)\}\), \(T_2 = \{(u, P_1, x_1, (x_1)_{n}, R_1, (y_1)_{n}, y_1, Q_1^{-1}, v)\}\).

Case 2. \(d(u, v) \geq 3\). By Lemma 9, there are \((n-2)\) paths \(P_1, P_2, \ldots, P_{n-2}\) of \(S_1^n\) such that (1) \(P_1\) joins \(u\) to a black node \(x_1\) with \((x_1)_{1} = 1\) and \(l(P_1) = (n-1)-(n-3)-1\), (2) \(P_2\) joins \(u\) to a white node \(x_2\) with \((x_2)_{1} = 1 + 1\) and \(l(P_2) = (n-1)-(n-3)\) for \(2 \leq i \leq n-2\), (3) \(\cup_{i=1}^{n-2}P_i\) spans \(S_1^n\), and (4) \(\cap_{i=1}^{n-2}V(P_i) = \{u\}\). Again, there are \((n-2)\) paths \(Q_1, Q_2, \ldots, Q_{n-2}\) of \(S_1^n\) such that (1) \(Q_1\) joins \(v\) to a black node \(y_1\) with \((y_1)_{1} = 1\) and \(l(Q_1) = (n-1)-(n-3)-1\), (2) \(Q_2\) joins \(v\) to a black node \(y_2\) with \((y_2)_{1} = 1 + 1\) and \(l(Q_2) = (n-1)-(n-3)\) for \(2 \leq i \leq n-2\), (3) \(\cup_{i=1}^{n-2}Q_i\) spans \(S_1^n\), and (4) \(\cap_{i=1}^{n-2}V(Q_i) = \{v\}\).

By Theorem 4, there exists a Hamiltonian path \(R_1\) of \(S_1^n\) joining the black node \((x)_{n-1}\) to the white node \((y)_{n-1}\) for every \(2 \leq i \leq n-2\) and \(i \neq t\). We set

\[
T_1 = \{(u, P_1, x_1, (x_1)_{n}, H_1, (y)_{n}, y_1, Q_1^{-1}, v)\},
\]

\[
T_2 = \{(u, P_1, x_1, (x_1)_{n}, R_1, (y_1)_{n}, y_1, Q_1^{-1}, v)\},
\]

\[
T_{n-1} = \{(u, (u)_{n}, H_2, (w)_{n}, w, (w)_{n}, v)\}.
\]

Obviously, \(T_1, T_2, \ldots, T_{n-1}\) is an \((n-1)\)-container of \(S_1^n\) between \(u\) and \(v\). Moreover, \(l(T_i) \leq (n-1)! + 2(n-2)! + 2(n-3)+1\). Thus, \(d_{n-1}^{-}\) is a \((n-1)! + 2(n-2)! + 2(n-3)+1\) Hamiltonian cycle of \(S_1^n\). Hence, this statement is proved.

Theorem 5.

\[
D_{n-1}^{(1)}(S_1^n) = \begin{cases} 
1 & \text{if } n = 2, \\
5 & \text{if } n = 3, \\
15 & \text{if } n = 4, \\
(n-1)! + 2(n-2)! + 2(n-3)+1 & \text{if } n \geq 5.
\end{cases}
\]

Proof. It is easy to check that \(D_{n-1}^{(1)}(S_2^n) = 1\) and \(D_{n-1}^{(1)}(S_3^n) = 5\). By Lemma 10, \(D_{n-1}^{(1)}(S_4^n) = 15\). By Lemmas 11, 12, and 13, we have \(D_{n-1}^{(1)}(S_n^n) = (n-1)! + 2(n-2)! + 2(n-3)+1\) if \(n \geq 5\). Hence, this statement is proved.

5. THE 2-\(d\)-DIAMETER \(S_n\)

Lemma 14. \(D_2^2(S_4^n) = 15\).
know \( d^v_{\text{in}}(1234, v) = 13 \) if \( v \in \{1423, 3142, 2413, 1243, 3421, 1432, 2341, 4123, 4312\} \) and \( d^v_{\text{in}}(1234, v) = 15 \) if \( v \in \{2134, 3214, 4231\} \). Hence, \( D^2_{\text{S}}(S_4) = 15 \).  

**Lemma 15.** Assume that \( a \) and \( b \) are any two distinct elements of \( S_4 \) and \( u \) is any white node of \( S_4 \). There exist two paths \( P_1 \) and \( P_2 \) of \( S_4 \) such that (1) \( P_1 \) joins \( u \) to a black node \( x \) with \( (x)_1 = a \) and \( l(P_1) = 5 \), (2) \( P_2 \) joins \( u \) to a white node \( y \) with \( (y)_1 = b \) and \( l(P_2) = 18 \), and (3) \( P_1 \cup P_2 \) spans \( S_4 \).

**Proof.** Because \( S_4 \) is node transitive, we may assume that \( u = 1234 \). The required two paths are listed below.

<table>
<thead>
<tr>
<th>( P_1 )</th>
<th>( P_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1234, 3214, 2431, 4123, 3412))</td>
<td>((1234, 4312, 2341, 1432, 3142))</td>
</tr>
<tr>
<td>((1234, 3214, 2431, 4123, 3412))</td>
<td>((1234, 4312, 2341, 1432, 3142))</td>
</tr>
<tr>
<td>((1234, 3214, 2431, 4123, 3412))</td>
<td>((1234, 4312, 2341, 1432, 3142))</td>
</tr>
</tbody>
</table>

Hence, this statement is proved. 

**Theorem 6.**

\[
D^2_{\text{S}}(S_n) = \begin{cases} 
5 & \text{if } n = 3, \\
15 & \text{if } n = 4, \\
\frac{n!}{2} + 1 & \text{if } n \geq 5.
\end{cases}
\]

**Proof.** It is easy to check that \( D^2_{\text{S}}(S_3) = 5 \). By Lemma 14, we have that \( D^2_{\text{S}}(S_4) = 15 \). Thus, we assume that \( n \geq 5 \). Let \( u \) be a white node and \( v \) be a black node of \( S_n \). Let \( P_1 \) and \( P_2 \) be any 2*-container of \( S_n \) joining \( u \) to \( v \). Obviously, \( \max\{l(P_1), l(P_2)\} \geq \frac{n!}{2} + 1 \). Hence, \( d^u_{\text{in}}(u, v) \geq \frac{n!}{2} + 1 \) and \( D^2_{\text{S}}(S_n) \geq \frac{n!}{2} + 1 \). Hence, we only need to show that \( d^u_{\text{out}}(u, v) \leq \frac{n!}{2} + 1 \). Because \( S_n \) is edge transitive, we assume that \( u \in S_n^{[n]} \) and \( v \in S_n^{[n-1]} \).

**CASE 1.** \( n = 5 \). By Lemma 15, there exist two paths \( H_1 \) and \( H_2 \) of \( S_5^{[5]} \) such that (1) \( H_1 \) joins \( u \) to a black node \( x \) with \( (x)_1 = 1 \) and \( l(H_1) = 5 \), (2) \( H_2 \) joins \( u \) to a white node \( y \) with \( (y)_1 = 3 \) and \( l(H_2) = 18 \), and (3) \( H_1 \cup H_2 \) spans \( S_5^{[5]} \). Again, there exist two paths \( T_1 \) and \( T_2 \) of \( S_4 \) such that (1) \( T_1 \) joins \( v \) to a white node \( p \) with \( (p)_1 = 2 \) and \( l(T_1) = 5 \), (2) \( T_2 \) joins \( v \) to a black node \( q \) with \( (q)_1 = 3 \) and \( l(T_2) = 18 \), and (3) \( T_1 \cup T_2 \) spans \( S_4^{[4]} \). By Lemma 3, there is a hamiltonian path \( R \) of \( S_4^{[1,2]} \) joining the white node \( (x)_1 \) to the black node \( (p)_1 \). Again, there is a hamiltonian path \( Z \) of \( S_4^{[1]} \) joining the black node \( (y)_1 \) to the white node \( (q)_1 \). We set

\[
L_1 = \langle u, H_1, x, (x)_1, R, (p)_1, p, T_1^{-1}, v \rangle \quad \text{and} \quad L_2 = \langle u, H_2, y, (y)_1, Z, (q)_1, q, T_2^{-1}, v \rangle.
\]

Obviously, \( L_1, L_2 \) is a 2*-container. Moreover, \( l(L_1) = 59 \) and \( l(L_2) = 61 \). Hence, \( D^2_{\text{S}}(u, v) \leq \frac{n!}{2} + 1 \). See Figure 8(a) for an illustration.

**CASE 2.** \( n \geq 6 \) is even. Let \( x \) be a neighbor of \( u \) in \( S_n^{[n]} \) with \( (x)_1 \in (n - 2) \). Let \( y \) be a neighbor of \( v \) in \( S_n^{[n-1]} \). Let \( z \) be a neighbor of \( y \) in \( S_n^{[n-1]} \) with \( (z)_1 \in (n - 2) \). Let \( a_1a_2 \ldots a_{n-2} \) be a permutation of \( (n - 2) \) such that \( a_1 = (x)_1 \) and \( a_{n-2} = (x)_1 \). Let \( H = \{a_1, a_2, \ldots, a_{n-2}\} \). By Theorem 2, there is a hamiltonian path \( P \) of \( S_n^{[n]} \) joining \( u \) to a white node \( p \) with \( (p)_1 = a_{n-2} \). By Theorem 4, there is a hamiltonian path \( Q \) of \( S_n^{[n-1]} \) joining a white node \( q \) with \( (q)_1 = a_1 \) to \( v \). By Theorem 3, there is a hamiltonian path \( R \) of \( S_n^{[2]} \) joining the white node \( (x)_1 \) to the black

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node (q)n. Again, there is a hamiltonian path W of S[n−2−H]
joining the black node (p)n to the white node (z)n. We set

\[ L_1 = (u, x, (x)^n, R, (q)^n, q, Q, v) \]

and

\[ L_2 = (u, P, (p)^n, M, (t)^n, W, (z)^n, z, y, v). \]

Obviously, \{L1, L2\} is a 2*n/container of S_n between u and v. Because l(L1) = \( n^2 - 2 \) and l(L2) = \( n^2 + 1 \), we have \( d_{2}\left( u, v \right) \leq \frac{n}{2} + 1 \). See Figure 8(b) for an illustration.

**Case 3.** n ≥ 7 is odd. Let x be a neighbor of u in S[n−1]n with (x)_1 \( \in (n-2) \). Let y be a neighbor of v in S[n−1]n. Let z be a neighbor of y in S[n−1]n with (z)_1 \( \in (n-2) - \{ (v), (y), (x)_1 \} \).

Let \( a_1a_2...a_{n-2} \) be a permutation of \( (n-2) \) such that \( a_1 = (x)_1 \) and \( a_{n-3} = (z)_1 \). Let \( H = \{a_1, a_2, ..., a_{n-2}\} \) and \( T = \{a_{n-1}, a_{n-2}, ..., a_2\} \). We set \( A = \{(i, a_{n-2}) | i \in H \cup \{n-1\}\} \) and \( B = \{(i, a_{n-2}) | i \in T \cup \{n\}\} \). Let \( S^A_t \) denote the subgraph of \( S_n \) induced by \( \cup_{i \in H \cup \{n-1\}} S^{(i,a_{n-2})}_n \) and let \( S^B_t \) denote the subgraph of \( S_n \) induced by \( \cup_{i \in T \cup \{n\}} S^{(i,a_{n-2})}_n \).

By Theorem 2, there is a hamiltonian path \( P \) of \( S^{(n-1)}_n - \{x\} \) joining u to a white node p with (p)_1 = a_{n-2} and (p)_n−3 = a_{n-3}. By Theorem 4, there is a hamiltonian path Q of \( S^{(n-3)}_n - \{y, z\} \) joining a white node q with (q)_1 = a_{n-2} and (q)_n−1 = a_1 to v. By Theorem 3, there is a hamiltonian path L of \( S^n_t \) joining a white node s with (s)_1 = a_1 to the black node (q)_n. Again, there is a hamiltonian path M of \( S^n_t \) joining the black node (p)_n to the white node (z)_n. We set

\[ L_1 = (u, x, (x)^n, R, (s)^n, s, L, (q)_n, q, Q, v) \]

and

\[ L_2 = (u, P, (p)^n, M, (t)^n, W, (z)_n, z, y, v). \]

Obviously, \{L1, L2\} is a 2*n/container of S_n between u and v. Because l(L1) = \( n^2 - 2 \) and l(L2) = \( n^2 + 1 \), we have \( d_{2}\left( u, v \right) \leq \frac{n}{2} + 1 \). See Figure 8(c) for an illustration.

**6. CONCLUSION**

In this study, we prove that \( D^{\infty}_{n-1}(S_n) = (n - 1)! + 2(n - 2)! + 2(n - 3)! + 1 = \frac{n!}{2} + 1 \) and \( D^{\infty}_{2}(S_n) = \frac{n!}{2} + 1 \) for \( n \geq 5 \). Actually, we prove that \( d_{2}\left( u, v \right) = \frac{n!}{2} + 1 \) for any two vertices u and v from different bipartite sets of \( S_n \).

Recently, we have proved that \( S_n \) is super laceable [19]. Hence, we can study \( D^{\infty}_{k}(S_n) \) for \( 1 \leq k \leq n - 1 \). We conjecture that \( D^{\infty}_{k}(S_n) = \frac{n!}{2} + 1 \) for \( n \geq 5 \) and \( 3 \leq k \leq n - 2 \).

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**REFERENCES**


G.J. Simmons, Almost all $n$-dimensional rectangular lattices are Hamilton-laceable, Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, 1978, pp. 649–661.