THREE-DIMENSIONAL CELLULAR NEURAL NETWORKS AND PATTERN GENERATION PROBLEMS

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This work investigates three-dimensional pattern generation problems and their applications to three-dimensional Cellular Neural Networks (3DCNN). An ordering matrix for the set of all local patterns is established to derive a recursive formula for the ordering matrix of a larger finite lattice. For a given admissible set of local patterns, the transition matrix is defined and the recursive formula of high order transition matrix is presented. Then, the spatial entropy is obtained by computing the maximum eigenvalues of a sequence of transition matrices. The connecting operators are used to verify the positivity of the spatial entropy, which is important in determining the complexity of the set of admissible global patterns. The results are useful in studying a set of global stationary solutions in various Lattice Dynamical Systems and Cellular Neural Networks.

Keywords: Three-dimensional Cellular Neural Networks; Lattice Dynamical Systems; spatial entropy; pattern generation; connecting operator.

1. Introduction


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In the one-dimensional case, spatial entropy $h$ can be determined exactly using an associated transition matrix $T$, i.e., $h = \log \lambda(T)$, where $\lambda(T)$ is the maximum eigenvalue of $T$.

For a two-dimensional situation, Ban and Lin [2005] developed a systematical approach for determining the high order transition matrix $T_n$. They determined the spatial entropy $h$ by computing the maximum eigenvalues of a sequence of such transition matrices. For a class of admissible local patterns, meaning for a class of $T_n$, the limiting equation of $\rho = \exp(\lambda(T))$ can be exactly solved using the recursive formula for $\rho(T_n)$. However, $T_n$ is a $2^n \times 2^n$ matrix, and $\rho(T_n)$ is usually quite difficult to compute for large $n$. The connecting operator and the trace operator have been derived to overcome these difficulties [Ban et al., 2007]; lower-bound estimates of entropy have been obtained by introducing connecting operators $C_{mn}$, and upper-bound estimates of entropy have been made by introducing trace operators $T_{mn}$.

This work develops a general method to investigate three-dimensional pattern generation problems, extending other studies [Ban & Lin, 2005] and [Ban et al., 2007] to the three-dimensional case. It focuses on ordering matrices of patterns and on the connecting operator in the three-dimensional case. The trace operator has been described elsewhere [Ban et al., 2008b]. This work is motivated by 3DCNN, so it is a major tool to study global patterns in 3DCNN.

Three-dimensional pattern generation problems are considered initially. Let $S$ be a finite set of $p \geq 2$ colors, where $Z^3$ denotes the integer lattice of $3D$. Denote, $U : Z^3 \rightarrow S$, a global pattern by $U(\alpha_1, \alpha_2, \alpha_3) = u_{\alpha_1,\alpha_2,\alpha_3}$. The set of all patterns with $p$ colors in a three-dimensional lattice is expressed as $\Sigma_p = \{U(U : Z^3 \rightarrow S)\}$. The set of all local patterns on $Z_{m_1 \times m_2 \times m_3}$ is denoted by

$$\Sigma_{m_1 \times m_2 \times m_3} = \{U(z_{m_1 \times m_2 \times m_3}) | U \in \Sigma_p\}$$

where $Z_{m_1 \times m_2 \times m_3} = \{(\alpha_1, \alpha_2, \alpha_3) | 1 \leq \alpha_i \leq m_i, 1 \leq i \leq 3\}$ is an $m_1 \times m_2 \times m_3$ finite rectangular lattice. For simplicity, two colors on the $2 \times 2 \times 2$ lattice $Z_{2 \times 2 \times 2}$ are considered here. Given a basic set $B \subset \Sigma_{2 \times 2 \times 2}$, the spatial entropy can be defined as

$$h(B) = \lim_{m_1, m_2, \ldots, m_3 \rightarrow \infty} \frac{\log \Gamma(m_1 \times m_2 \times m_3(B))}{m_1 m_2 m_3}$$

(1)

where $\Gamma(m_1 \times m_2 \times m_3(B))$ is the number of distinct patterns in $\Sigma_{m_1 \times m_2 \times m_3}$ and $\Gamma(m_1 \times m_2 \times m_3)$ is the set of all local patterns on $Z_{m_1 \times m_2 \times m_3}$, which can be generated from $B$, as described elsewhere [Chow et al., 1996b]. Six different orderings

\begin{align*}
\end{align*}

are obtained and the ordering matrix $W_{2 \times 2 \times 2}$ for $\Sigma_{2 \times 2 \times 2}$ can be introduced according to the different ordering $[z]$. Without loss of generality, $X_{2 \times 2 \times 2}$ is considered and the other cases are similar.

One of the main results is the construction of $X_{2 \times 2 \times 3}$ from $X_{2 \times 2 \times 2}$, where $X_{2 \times 2 \times 3}$ represents the ordering matrix of $\Sigma_{2 \times 2 \times 3}$ according to $[z]$-ordering. It can be addressed in the following three steps.
Step I. Apply \( [z] \)-ordering to \( Z_{1 \times m_2 \times 2} \)

<table>
<thead>
<tr>
<th>2</th>
<th>4</th>
<th>...</th>
<th>2k</th>
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<tr>
<td>1</td>
<td>3</td>
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<td>2k−1</td>
<td>...</td>
<td>( m_2 )</td>
<td>( m_2 )−1</td>
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and introduce ordering matrix \( X_{2 \times m_2 \times 2} \) for \( \Sigma_{2 \times m_2 \times 2} \) as in Theorem 2.1. By Theorem 3.2, the transition matrix \( \hat{A}_{2 \times m_2 \times 2} \) can be obtained from

\[
A_{2 \times 2 \times 2 \times 2} = (A_{2 \times (m_1-1) \times 2} \otimes (m_2-1 \times 2^{m_2-1}) \otimes (E_{2^{m_2-1}}) \otimes A_{2 \times 2 \times 2}^t)
\]

where \( E_2 \) is the \( 2 \times 2 \) matrix with 1 as its entries, \( \otimes \) is the tensor product and \( \circ \) is the Hadamard product, as in Eq. (36).

Step II. Convert \([x]\)-ordering into \([z]\)-ordering on \( Z_{1 \times m_2 \times 2} \) using

\[
s = \begin{pmatrix} m_1+1 & \ldots & \ldots & m_2 \end{pmatrix}
\]

and introduce the ordering matrix \( \hat{X}_{2 \times m_2 \times 2} \) for \( \Sigma_{2 \times m_2 \times 2} \) as in Theorem 2.4. The associated transition matrix \( \hat{A}_{2 \times m_2 \times 2} \) is given by

\[
\hat{A}_{2 \times 2 \times 2 \times 2} = \hat{F}_{2 \times 2 \times 2 \times 2} A_{2 \times 2 \times 2 \times 2} \hat{F}_{2 \times 2 \times 2 \times 2}
\]

where \( \hat{F}_{2 \times 2 \times 2 \times 2} \) is the permutation matrix as in Theorem 3.4.

Step III. Define \([x]\)-ordering on \( Z_{1 \times m_2 \times 2} \) as

\[
s = \begin{pmatrix} m_1+1 \times m_2 \times \ldots & \ldots & m_2 \end{pmatrix}
\]

and introduce ordering matrix \( \hat{X}_{2 \times m_2 \times m_3} \) for \( \Sigma_{2 \times m_2 \times m_3} \) as in Theorem 2.5. The recursive formula for the transition matrix \( \hat{A}_{2 \times m_2 \times m_3} \) can be obtained by

\[
\hat{A}_{2 \times 2 \times m_3 \times m_2} = (A_{2 \times 2 \times m_3 \times m_2} (m_1-1) \otimes V_{2 \times m_2 \times m_3} \otimes \hat{A}_{2 \times 2 \times m_3 \times m_2})
\]

as in Theorem 3.5.

Theorem 3.7 enables the maximum eigenvalue \( \lambda_{2 \times m_2 \times m_3} \) of \( \hat{A}_{2 \times m_2 \times m_3} \) to be computed, to yield the spatial entropy,

\[
h(S) = \lim_{m_2 \to \infty} \frac{1}{m_2} \log \lambda_{2 \times m_2 \times m_3}.
\]

However, some estimates of lower bound of spatial entropy \( h(S) \) can be made using the connecting operator. Then, for fixed \( m_1, m_2 \geq 2 \), the \( m_3 \)-limit in Eq. (1) is studied:

\[
\lim_{m_3 \to \infty} \frac{1}{m_3} \log \left| A_{2 \times m_2 \times m_3} \right|.
\]

The recursive formula of \( A_{2 \times m_2 \times m_3} \) in \( m_3 \) is considered. Accordingly, the next task is to investigate Eq. (3). According to Eqs. (53) and (54),

\[
A^m_{2 \times m_2 \times m_3} = \sum_{k=1}^{2m_2(m_3-1)} A^{(k)}_{2 \times m_2 \times m_3}
\]

where \( A^{(k)}_{2 \times m_2 \times m_3} \) is called an elementary pattern of order \((m_1, m_2, m_3)\) and is a fundamental element in constructing \( A_{2 \times m_2 \times m_3} \). \( V_{2 \times m_2 \times m_3} \) is defined as

\[
V_{2 \times m_2 \times m_3} = [V_{2 \times m_2 \times m_3} (x_1) | \ldots | V_{2 \times m_2 \times m_3} (x_{m_3})]
\]

as in Eqs. (55) and (56), which specifies systematically these elementary patterns. The connecting operator \( C_{2 \times m_2 \times m_3} \) is introduced as in Definition 4.2, and used to derive a recursive formula for \( A^{(k)}_{2 \times m_2 \times m_3} \) as in Theorem 4.5

\[
V_{2 \times m_2 \times m_3} = C_{2 \times m_2 \times m_3} (x_{m_3-1}) V_{2 \times m_2 \times m_3}
\]

where \( C_{2 \times m_2 \times m_3} = S_{2 \times m_2 \times m_3} \). The recursive formula Eq. (67) yields a lower bound on entropy

\[
h(A_{2 \times 2 \times 2}) \geq \lim_{m_3 \to \infty} \frac{1}{m_2 m_3} \log \left| S_{2 \times m_2 \times m_3} \right|
\]
such as in Theorem 4.12 and which implies $h(A_{x_{2}x_{2}x_{2}}) > 0$ if a diagonal periodic cycle is applied with a limit in Eq. (4) that exceeds 0. This method powerfully yields the positivity of spatial entropy, which is useful in evaluating the complexity of patterns generation problems.

The method is very effective in elucidating the complexity of the set of mosaic patterns in 3DCNN. A typical 3DCNN is of the form

$$\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + \sum_{|\alpha|,|\beta|,|\gamma| \leq 1} a_{\alpha,\beta,\gamma} f(u_{i+\alpha,j+\beta,k+\gamma}) + \sum_{|\alpha|,|\beta|,|\gamma| \leq 1} b_{\alpha,\beta,\gamma} w_{i+\alpha,j+\beta,k+\gamma}$$

(5)

where $(i,j,k) \in \mathbb{Z}^{3}$, $f(u)$ is a piecewise-linear output function, defined by

$$v = f(u) = \frac{1}{2}(|u| + |u - 1|).$$

Here, $A = (a_{\alpha,\beta,\gamma})$ is a feedback template, a spatial-invariant template; $B = (b_{\alpha,\beta,\gamma})$ is a controlling template, and $w$ is called a biased term or threshold. To elucidate the method, consider nonzero $a_{0,0,0} = a$, $a_{1,0,0} = a$, $a_{0,1,0} = a$, $a_{0,0,1} = a$, and zero other $a_{\alpha,\beta,\gamma}$ and $b_{\alpha,\beta,\gamma}$. Therefore, Eq. (5) can be rewritten as

$$\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + af(u_{i,j,k}) + af(u_{i+1,j,k}) + af(u_{i,j+1,k}) + af(u_{i,j,k+1}).$$

(6)

The quantities $u_{i,j,k}$ represent the state of cell at $(i,j,k)$. The stationary solution $\bar{u} = (\bar{u}_{i,j,k})$ of Eq. (6) satisfies

$$u_{i,j,k} = w + a_{i,j,k} + a_{i+1,j,k} + a_{i,j+1,k} + a_{i,j,k+1},$$

(7)

where $v = f(u)$, which is very important in studying 3DCNNs: their outputs $\bar{v} = (\bar{v}_{i,j,k}) = f(\bar{u}_{i,j,k})$ are called patterns. A mosaic solution $\bar{u}$ satisfies $|\bar{u}_{i,j,k}| \geq 1$ and its corresponding pattern $\bar{v}$ is called a mosaic pattern where $|\bar{u}_{i,j,k}| \geq 1$ for all $(i,j,k) \in \mathbb{Z}^{3}$. Among the stationary solutions, the mosaic solutions are stable and are crucial to study the complexity of Eq. (6). Equation (7) has five parameters $w, a_{x, y, z}$ and $a_{x, y} < a_{y, z} < a_{x, z} < 0$ and $|a_{x}| > |a_{y}| + |a_{z}|$ are considered to elucidate application of our work. In particular, region [4, 8] in Fig. 4 in Sec. 5 is considered: the transition matrix can be written as

$$A_{x_{2}x_{2}x_{2}} = G \otimes E \otimes E \otimes E,$$

where $G = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$ and $E = \left[ \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right]$.

Then, Steps I–III yield the aforementioned admissible patterns in $G_{x_{2}x_{2}x_{2}}$; the corresponding transition matrix can be derived as in Proposition 3.9.

Step I $\Rightarrow A_{x_{2}x_{2}x_{2}} = (G \otimes E)^{m_{1}-1} \otimes (\otimes E)$,

Step II $\Rightarrow A_{x_{2}x_{2}x_{2}} = (G^{m_{1}-1} \otimes (\otimes E^{m_{2}}),$ 

Step III $\Rightarrow A_{x_{2}x_{2}x_{2}} = (G^{m_{1}-1} \otimes E^{m_{2}-1} \otimes (\otimes E^{m_{3}}).$

The complexity of the 3DCNN model, as in Eq. (6), can be examined using the connecting operator defined in Sec. 4. Since the connecting operator $C_{x_{1}m_{1}m_{2}m_{3}} = S_{x_{1}m_{1}m_{2}m_{3}} \ × (\otimes E^{m_{1}-1} \ × E),$ the maximum eigenvalue can be exactly computed as

$$\rho(S_{x_{1}m_{1}m_{2}m_{3}}) = 2g^{m_{2}-1},$$

where $g = (1 + \sqrt{5})/2$ is the golden-mean, as in Proposition 5.1. According to Eq. (4), the lower bound of spatial entropy in the region (VIII)-(i)-(1)-(1)-(4,8) can be estimated

$$h(A_{x_{2}x_{2}x_{2}}) \geq \lim_{m_{1} \to \infty} \frac{1}{2m_{2}} \log \rho(S_{x_{1}m_{1}m_{2}m_{3}}) = \frac{1}{2} \log g.$$

Moreover, in this case, spatial entropy can be solved exactly from the maximum eigenvalue of $A_{x_{2}x_{2}x_{2}}$. Since

$$\rho(A_{x_{2}x_{2}x_{2}}) = 2m_{1}m_{2}^{-1}g^{m_{1}-1}(m_{1}-1),$$

the spatial entropy is

$$h(A_{x_{2}x_{2}x_{2}}) = \lim_{m_{2} \to \infty} \frac{\rho(A_{x_{2}x_{2}x_{2}})}{m_{2}m_{3}} = \log g$$

as in Proposition 3.9.

The rest of this paper is organized as follows. Section 2 derives a recursive formula for the ordering matrix $X_{2x_{2}x_{2}}$ for $X_{2x_{2}x_{2}}$ from $X_{2x_{2}x_{2}}$. The ordering $[x]$ is converted to $[\bar{x}]$. Then, a similar recursive formula is constructed for ordering matrix $X_{2x_{2}x_{2}}$ from $X_{2x_{2}x_{2}}$. Section 3 derives the recursive formula for the associated high order transition matrices $A_{x_{2}x_{2}x_{2}}$. Then, Section 4 derives the connecting operator $C_{x_{1}m_{1}m_{2}m_{3}}$, which can recursively reduce elementary patterns of high order to patterns of low

order. Then, the lower-bound of spatial entropy is determined by computing the maximum eigenvalues of the diagonal periodic cycles of sequence \( S_{x,m_1,m_2,m_3} \). Section 5 gives an example of the application of our main results to 3DCNN.

2. Three-Dimensional Pattern Generation Problems

This section describes three-dimensional pattern generation problems. Here, \( m_1, m_2, m_3 \geq 2 \) are fixed and indices are omitted for brevity. Let \( S \) be a set of \( p \) colors, and \( Z_{m_1 \times m_2 \times m_3} \) be a fixed finite rectangular sublattice of \( \mathbb{Z}^3 \), where \( \mathbb{Z}^3 \) denotes the integer lattice on \( \mathbb{R}^3 \) and \((m_1, m_2, m_3) \) a three-tuple of positive integers. Functions \( U : \mathbb{Z}^3 \rightarrow S \) and \( U_{m_1 \times m_2 \times m_3} : Z_{m_1 \times m_2 \times m_3} \rightarrow S \) are called global patterns and local patterns on \( Z_{m_1 \times m_2 \times m_3} \), respectively. The set of all patterns \( U \) is denoted by \( \Sigma^3 \subseteq \mathcal{S}^3 \) such that \( \Sigma^3 \) is the set of all patterns with \( p \) different colors in a three-dimensional lattice. For clarity, two symbols, \( S = \{0, 1\} \) are considered. Let \( x, y \) and \( z \) coordinate represent the 1st-, 2nd- and 3rd-coordinates respectively as in Fig. 1.

Six orderings \( \omega \) ordering are represented as Eq. (8)

\[
\begin{align*}
[z] : [3] > [2] > [1]
\end{align*}
\] (8)

![Fig. 1. Three-dimensional coordinate system.](Image)

On a fixed finite lattice \( Z_{m_1 \times m_2 \times m_3} \), an ordering \( \omega \) : \([x] > [y] > [z]\) is obtained on \( Z_{m_1 \times m_2 \times m_3} \), which is any one of the above orderings on \( Z_{m_1 \times m_2 \times m_3} \).

\[
\psi_\omega(a_1, a_2, a_3) = m_1 m_2 (a_1 - 1) + m_2 (a_2 - 1) + a_3,
\]

where \( 1 \leq a_1 \leq m_1 \) and \( 1 \leq \ell \leq 3 \). The ordering \( \omega \) on \( Z_{m_1 \times m_2 \times m_3} \) can now be applied to \( \Sigma_{m_1 \times m_2 \times m_3} \). Indeed, for each \( U = (u_{a_1 a_2 a_3}) \in \Sigma_{m_1 \times m_2 \times m_3} \), define

\[
\psi_\omega(U) \equiv \psi_{\omega; m_1, m_2, m_3}(U)
\]

\[
\equiv 1 + \sum_{a_1=1}^{m_1} \sum_{a_2=1}^{m_2} \sum_{a_3=1}^{m_3} u_{a_1 a_2 a_3} \omega_{a_1 a_2 a_3}
\]

where

\[
\omega_{a_1 a_2 a_3} = 2^m m_1 m_2 - \psi(a_1, a_2, a_3)
\]

\[
= 2^m m_1 (m_1 - a_1) + m_2 (m_2 - a_2) + (m_3 - a_3).
\]

\( U \) is referred to herein as the \( \psi_\omega(U) \)-th element in \( \Sigma_{m_1 \times m_2 \times m_3} \) by ordering \( \omega \). Identifying the pictorial patterns using \( \psi_\omega(U) \) is very effective in proving theorems since computations can now be performed on \( \psi_\omega(U) \). For instance, the orderings on \( \mathbb{Z}_{2 \times 2 \times 2} \) can be represented as in Fig. 2.

2.1. Ordering matrices

The cube \( Z_{m_1 \times m_2 \times m_3} \) can be decomposed by \( m_1 \)-many \((m_2\text{-many and } m_3\text{-many})\) parallel two-dimensional rectangles in \( Z_{1 \times m_2 \times m_3} \) (\( Z_{m_1 \times 1 \times m_3} \) and \( Z_{m_1 \times m_2 \times 1} \)). Any patterns \( U = (u_{a_1 a_2 a_3}) \in \Sigma_{m_1 \times m_2 \times m_3} \) can be decomposed accordingly. For example, in \([x]\)-ordering, define the \( a_1 \)th layer of rectangle as

\[
Z_{a_1, m_2 \times m_3} = \{(a_1, a_2, a_3)| 1 \leq a_2 \leq m_2, 1 \leq a_3 \leq m_3 \}.
\]

Pattern \( U \) in \( a_1 \)th layer is assigned the number

\[
i_{a_1} = 1 + \sum_{a_2=1}^{m_2} \sum_{a_3=1}^{m_3} u_{a_1 a_2 a_3} x_{1, m_2, m_3},
\]

where \( x_{1, m_2, m_3} = 2^m m_2 - m_2 (a_2 - 1) - a_3. \) As denoted by the \( 1 \times m_2 \times m_3 \) pattern

\[
x_{1 \times m_2 \times m_3, i_{a_1}} =
\]

\[
\begin{bmatrix}
u_{a_1 1 m_2} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
u_{a_1 m_2 1} & \cdots & \cdots & \cdots \\
u_{a_1 1 2} & \cdots & \cdots & \cdots \\
u_{a_1 2 1} & \cdots & \cdots & \cdots \\
u_{a_1 m_2 2}
\end{bmatrix}
\]
In particular, when \( m_2 = 2 \) and \( m_3 = 2 \), as denoted by \( x_{1 \times 2 \times 2 \times i_1} \), where

\[
i_{\alpha_1} = 1 + 2^3 u_{\alpha_111} + 2^2 u_{\alpha_112} + 2 u_{\alpha_121} + u_{\alpha_122}
\]

and

\[
x_{1 \times 2 \times 2 \times i_1} \equiv x_{i_1} = \begin{pmatrix} u_{\alpha_111} & u_{\alpha_112} \\ u_{\alpha_121} & u_{\alpha_122} \end{pmatrix},
\]

where \( \alpha_1 \in \{1, 2\} \). A \( 2 \times 2 \times 2 \) pattern \( U = (u_{\alpha_1 \alpha_2 \alpha_3}) \) can now be obtained from the \([x]\)-direct sum of two \( 1 \times 2 \times 2 \) patterns using \([x]\)-ordering:

\[
x_{2 \times 2 \times 2 \times i_1 i_2} \equiv x_{i_1 i_2} \equiv x_{i_1} \oplus x_{i_2}
\]

where \( i_{\alpha_1} \) as in Eq. (10) and \( \alpha_1 \in \{1, 2\} \). Therefore, the complete set of \( 2^8 \) patterns in \( \Sigma_{2 \times 2 \times 2} \) is given by a \( 16 \times 16 \) matrix \( X_{2 \times 2 \times 2} = [x_{i_1 i_2}] \) as its entries in

\[
x_{i_1 i_2} = \begin{pmatrix}
\text{top} & \text{top} & \text{top} & \text{top} \\
\text{top} & \text{top} & \text{top} & \text{top}
\end{pmatrix},
\]

where \( x_{i_1 i_2} \) to substitute \( x_{i_1 \oplus i_2} \) for simplicity afterward.

---

2 Use \( x_{i_1 i_2} \) to substitute \( x_{i_1 \oplus i_2} \) for simplicity afterward.
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That
\[ \psi_i(x_{i,j,k}) = 2^i(i_1 - 1) + i_2 \]
is easily verified, and local patterns in \( \Sigma_{2 \times 2 \times 2} \) are then counted by going through each row successively in Eq. (11). Correspondingly, \( \Sigma_{2 \times 2 \times 2} \) can be referred to as an ordering matrix for \( \Sigma_{2 \times 2 \times 2} \). A \( 2 \times 2 \times 2 \) pattern can also be regarded as an \([x]\)-direct sum of two \( 1 \times 2 \times 2 \) patterns using \([z]\)-ordering,

\[ Z_{2 \times 2 \times 2} \equiv Z_{i,j,k} \]

where
\[ \hat{i}_{a1} = 1 + 2^3u_{a1,11} + 2^2u_{a1,21} + 2u_{a1,12} + u_{a1,22}, \quad a_1 \in \{1, 2\}. \]

The ordering matrix \( Z_{2 \times 2 \times 2} \) can be represented as

Now,
\[ \psi_i(x_{i,j,k}) = 2^i(i_1 - 1) + i_2 \]
can be verified. Similarly, a \( 2 \times 2 \times 2 \) pattern can also be viewed as a \([y]\)-direct ([\(j]\)-direct) and \([z]\)-direct ([\(i]\)-direct) sum of \( 2 \times 1 \times 2 \) and \( 2 \times 2 \times 1 \) patterns:

\[ y_{j_1,j_2} \equiv y_{j_1,j_2} \oplus y_{j_1,j_2}, \quad j_{1,2} \equiv j_{1,2} \oplus j_{1,2}, \quad \hat{x}_{k_1,k_2} \equiv \hat{x}_{k_1,k_2} \oplus \hat{x}_{k_1,k_2}, \quad \hat{i}_{k_1,k_2} \equiv \hat{i}_{k_1,k_2} \oplus \hat{i}_{k_1,k_2}, \quad \hat{z}_{k_1,k_2} \equiv \hat{z}_{k_1,k_2} \oplus \hat{z}_{k_1,k_2}. \]

where
\[ j_{a1} = 1 + 2^3u_{a1,11} + 2^2u_{a1,21} + 2u_{a1,12} + u_{a1,22}, \quad a_2 \in \{1, 2\}, \]
\[ j_{a1} = 1 + 2^3u_{a1,11} + 2^2u_{a1,21} + 2u_{a1,12} + u_{a1,22}, \quad a_2 \in \{1, 2\}, \]
\[ k_{a2} = 1 + 2^3u_{a1,11} + 2^2u_{a1,21} + 2u_{a1,12} + u_{a1,22}, \quad a_2 \in \{1, 2\}, \]
\[ k_{a2} = 1 + 2^3u_{a1,11} + 2^2u_{a1,21} + 2u_{a1,12} + u_{a1,22}, \quad a_3 \in \{1, 2\}. \]

A \( 16 \times 16 \) matrix \( Z_{2 \times 2 \times 2} = \{y_{j_1,j_2}\} \) or \( Z_{2 \times 2 \times 2} = \{z_{k_1,k_2}\} \) can also be obtained for \( \Sigma_{2 \times 2 \times 2} \), such that \( Z_{2 \times 2 \times 2} = \)

or \( Z_{2 \times 2 \times 2} \)
The relationship between $W_{2,2,2}$ must be studied, where $W \in \{X, Y, Z, \hat{X}, \hat{Y}, \hat{Z}\}$. Before the relations are explained, the column matrix and the row matrix must be given. Let $A = [a_{ij}]$ be an $m^2 \times m^2$ matrix, the column matrix $A^{(c)}$ of $A$ is defined as

$$A^{(c)} = \begin{bmatrix}
A_1^{(c)} & A_2^{(c)} & \cdots & A_m^{(c)} \\
A_{m+1}^{(c)} & A_{m+2}^{(c)} & \cdots & A_{2m}^{(c)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{(m-1)m+1}^{(c)} & A_{(m-1)m+2}^{(c)} & \cdots & A_{m^2}^{(c)}
\end{bmatrix},$$

where $1 \leq \alpha \leq m^2$.

The row matrix $A^{(r)}$ of $A$ is defined as

$$A^{(r)} = \begin{bmatrix}
A_1^{(r)} & A_2^{(r)} & \cdots & A_m^{(r)} \\
A_{m+1}^{(r)} & A_{m+2}^{(r)} & \cdots & A_{2m}^{(r)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{(m-1)m+1}^{(r)} & A_{(m-1)m+2}^{(r)} & \cdots & A_{m^2}^{(r)}
\end{bmatrix},$$

(13)

where $1 \leq \alpha \leq m^2$. Hence, based on some observations, $X_{2,2,2}$ can be represented in terms of $y_{2,2,2}$ as

$$X_{2,2,2} = \gamma_{2,2,2}.$$

Furthermore, $Y_{2,2,2} = \gamma_{2,2,2}^{(r)}$, $Z_{2,2,2} = \gamma_{2,2,2}^{(c)}$. $Y_{2,2,2}$ and $Z_{2,2,2}$ can also be obtained. The remainder of this subsection addresses the construction of $\hat{X}_{2,m^2 \times m^2}$ from $X_{2,2,2}$ in the following three steps, where $\hat{X}_{2,m^2 \times m^2}$ represents the ordering matrix of $\Sigma_2 \times m^2$ according to $[\hat{x}]$-ordering generated from $X_{2,2,2}$.

**Step I.** Apply $[\hat{x}]$-ordering to $Z_{1 \times m_2 \times 2}$ using

$$2 \quad 4 \quad \cdots \quad 2k \quad 2m_2$$

$$1 \quad 3 \quad \cdots \quad 2k-1 \quad 2m_2-1$$

(16)

and introduce ordering matrix $X_{2 \times m_2 \times 2}$ for $\Sigma_2 \times m_2 \times 2$.

**Step II.** Convert $[\hat{x}]$-ordering into $[\hat{z}]$-ordering on $Z_{1 \times m_2 \times 2}$ by

$$\begin{array}{cccc}
m_2+1 & m_2+2 & \cdots & 2m_2-1 & 2m_2 \\
1 & 2 & \cdots & k & m_2 \\
\end{array}$$

(17)

and introduce ordering matrix $\hat{X}_{2 \times m_2 \times 2}$ for $\Sigma_2 \times m_2 \times 2$.

**Step III.** Define $[\hat{z}]$-ordering on $Z_{1 \times m_2 \times m_2}$ by

$$\begin{array}{cccc}
m_2+1 & m_2+2 & \cdots & 2m_2-1 & 2m_2 \\
1 & 2 & \cdots & m_2 & m_2 \\
\end{array}$$

(18)

and introduce ordering matrix $\hat{X}_{2 \times m_2 \times m_2}$ for $\Sigma_2 \times m_2 \times m_2$.

To introduce $X_{2 \times m_2 \times 2}$, define

$$y_{2 \times m_2 \times 2} = \gamma_{2 \times m_2 \times 2}^{(r)} \oplus \gamma_{2 \times m_2 \times 2}^{(c)},$$

(19)

where $1 \leq m_2 \leq 2^k$ and $1 \leq k \leq m_2$. Herein, a wedge direct sum $\oplus$ is applied to $2 \times 2 \times 2$ patterns whenever they can be attached to each other.
Now, \( X_{2 \times m_2} \) can be obtained as follows.

**Theorem 2.1.** For any \( m_2 \geq 2 \), \( \Sigma_{2 \times m_2} = \{ y_{ij} \}_{i=1}^{j} \), where \( y_{ij} \) is given in Eq. (14). Furthermore, the ordering matrix \( X \) is obtained by decomposing into the two formulae, 

\[
X_{2 \times m_2} = \left[ X_{2 \times m_2}^{1 \times j} \right] \times 2^{j_1} \times 2^{j_2} \times \cdots \times 2^{j_k},
\]

where \( 1 \leq j_1 \leq 2^{k}. \) For fixed \( j_1, j_2, \ldots, j_k \in \{ 1, 2, \ldots, 2^k \} \),

\[
(X_{2 \times m_2})_{2 \times j_1} = [X_{2 \times m_2}^{2 \times j_2-j_1-i}] 	imes 2^{j_2-j_1-i},
\]

where \( 1 \leq j_1 \leq 2^k \) and \( k \in \{ 1, 2, \ldots, m_2 - 2 \}, \) for fixed \( j_1, j_2, \ldots, j_{m_2 - 2} \).

\[
(X_{2 \times m_2})_{2 \times j_1} = [y_{2 \times m_2}^{2 \times j_1}] 	imes 2^{j_1-i},
\]

where \( y_{2 \times m_2}^{2 \times j_1} \) is defined as in Eq. (19).

**Proof.** From Eq. (12), \( u_1(\alpha_0) \) can be solved in terms of \( j_2 \), yielding

\[
\begin{align*}
\text{For } j_1 &:= 2^{m_2} - 1, \\
\text{and } \alpha_0 &:= 2^{m_2 - 1} - 2, \\
\alpha &:= 2^{m_2 - 1} - 2 u_1(\alpha_0).
\end{align*}
\]

\[
\text{Next, } \alpha \text{-ordering is converted into } \hat{\alpha} \text{-ordering for } Z_{1 \times m_2} \text{.}
\]

In Eq. (17), the position of \((1, \alpha_2, \alpha_3)\) is the 0th, where

\[
\hat{\alpha} = m_2(\alpha_3 - 1) + \alpha_2.
\]

The relation

\[
\hat{\alpha} = m_2 \alpha + (1 - 2m_2) \frac{\alpha - 1}{2} + (1 - m_2),
\]

or

\[
\hat{\alpha} = m_2 + k \text{ if } \alpha = 2k - 1,
\]

and

\[
\hat{\alpha} = m_2 + k \text{ if } \alpha = 2k,
\]

where \( 1 \leq k \leq m_2 \) is easily verified.

Now, the ordering \( \hat{\alpha} \) in Eq. (17) on \( Z_{1 \times m_2} \) can be extended to \( Z_{1 \times m_2} \) by Eq. (18). For a fixed \( m_2 \), \( \hat{\alpha} \)-ordering on \( Z_{1 \times m_2} \) is clearly one-dimensional; it grows in the \( z \)-direction. Given
ordering $\mathbf{Z}_{i}^{\alpha_{1}}$, for $U = (u_{i}^{a_{2}}) \in \Sigma_{2}^{m_{2}}$, denoted by

$$i_{a_{2}} = 1 + \sum_{a_{1}=1}^{m_{2}} u_{i}^{a_{2}} 2^{m_{2}(m_{1}-a_{1})+(m_{2}-a_{2})},$$

where $\alpha_{1} = 1, 2$, \(\psi_{k}(U) = 2^{m_{2}m_{3}}(\bar{i}_{1} - 1) + \bar{i}_{2}\).

Now, let $\bar{x}_{i_{1}^{a_{2}}} = U = (u_{i}^{a_{2}})$, yielding the new ordering matrix $\mathbf{Z}_{i}^{\alpha_{1}} = \bar{x}_{i}^{a_{1}}$, for $\Sigma_{2}^{m_{2}}$. The relationship between $\mathbf{Z}_{i}^{\alpha_{1}}$ and $\mathbf{Z}_{i}^{\alpha_{2}}$ is established before $\mathbf{Z}_{i}^{\alpha_{1}}$ is constructed from $\mathbf{Z}_{i}^{\alpha_{2}}$ for $m_{2} \geq 3$.

A conversion sequence of orderings can be obtained from Eqs. (16) and (17), where $P_{k}$ represents the permutation of $\mathbb{N}_{2n} = \{1, 2, \ldots, 2m_{2}\}$ such that $P_{k}(k+1) = k$, $P_{k}(k) = k+1$ and the other numbers are fixed. $P_{k}$ is also the permutation on $\mathbf{Z}_{i}^{\alpha_{1}}$ such that it exchanges $k$ and $k+1$ while keeping the other positions fixed, i.e.

\[
\begin{pmatrix}
1 & 2 & \cdots & k-1 & k & k+1 & \cdots \n\end{pmatrix}
\]

Clearly, Eq. (16) can be converted into Eq. (17) in many ways using the sequence of $P_{k}$. A systematic approach is proposed here.

**Lemma 2.3.** For $m_{2} \geq 2$, Eq. (16) can be converted into Eq. (17) using the following sequences of $(m_{2}(m_{2} - 1))/2$ permutations successively

\[
(P_{k}P_{k+1} \cdots P_{2m_{2}-2})(P_{k}P_{k+1} \cdots P_{2m_{2}-3}) \cdots (P_{k}P_{k+1} \cdots P_{m_{2}+1})P_{m_{2}},
\]

where $2 \leq k \leq m_{2}$.

**Proof.** When $m_{2} = 2$ and 3, verifying that Eq. (25) can convert Eq. (16) into Eq. (17) is relatively simple.

When $m_{2} \geq 4$, and for any $2 \leq k \leq m_{2}$, applying

\[
(P_{k}P_{k+1} \cdots P_{2m_{2}-2})(P_{k}P_{k+1} \cdots P_{2m_{2}-3}) \cdots (P_{k}P_{k+1} \cdots P_{2m_{2}-k})
\]

to Eq. (16), yields two intermediate cases:

**Case (i):** When $2 \leq k \leq [m_{2}/2]$, \n
\[
\begin{pmatrix}
k+1 & k+2 & \cdots & k+1 & \cdots & \cdots & 2m_{2} \n1 & 2 & \cdots & k & k+1 & \cdots & 2m_{2}-k \n\end{pmatrix}
\]

where $0 \leq \ell \leq m_{2} - 2k$.

**Case (ii):** When $[m_{2}/2] + 1 \leq k \leq m_{2} - 1$,

\[
\begin{pmatrix}
k+1 & \cdots & k+1 & \cdots & \cdots & \cdots & 2m_{2} - 1 & 2m_{2} \n1 & 2 & \cdots & \cdots & \cdots & k-1 & k & 2m_{2} - k \n\end{pmatrix}
\]

When $k = m_{2}$ in Eq. (27), Eq. (17) holds. Equations (26) and (27) are established by mathematical induction on $k$. When $k = 2$, verifying that Eq. (16) is converted into

\[
\begin{pmatrix}
1 & 2 & 3 & \cdots & \cdots & \cdots & 2m_{2} - 3 & 2m_{2} - 1 & 2m_{2} \n1 & 2 & 4 & \cdots & \cdots & \cdots & 2m_{2} - 3 & 2m_{2} - 2 \n\end{pmatrix}
\]

by $P_{2}P_{1} \cdots P_{2m_{2}-2}$ is relatively easy such that Eq. (26) holds for $k = 2$. Next, assume that Eq. (26) holds for $k \leq [m_{2}/2]$. Then, by applying $P_{k+1}P_{k+3} \cdots P_{2m_{2}-k-1}$ to Eq. (26), Eq. (26) can be verified to hold for $k+1$ when $k+1 \leq [m_{2}/2]$ or becomes Eq. (27) when $k+1 \geq [m_{2}/2]$. When $k \geq [m_{2}/2] + 1$, $P_{k+1}P_{k+3} \cdots P_{2m_{2}-k-1}$ is applied to Eq. (27). Equation (27) can also be confirmed to
hold for \( k + 1 \). Finally, Eq. (17) is concluded to hold for \( k = m_2 \). The proof is thus complete.

Based on Lemma 2.3, \( X_{2 \times m_2 \times 2} \) can be converted into \( \hat{X}_{2 \times m_2 \times 2} \) as follows. Let

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

and for \( 2 \leq j \leq 2m_2 - 2 \), as denoted by

\[
P_{2m_2,j} = I_{2 \times j-1} \otimes P \otimes I_{2 \times 2m_2-j-1},
\]

where \( I_k \) is the \( k \times k \) identity matrix. Moreover, let

\[
\begin{align*}
\hat{P}_{2 \times m_2 \times 2} &= \left( P_{2m_2,0} \cdots P_{2m_2,4} \cdots P_{2m_2,2m_2-j} \right) \\
&\cdots \left( P_{2m_2,k} \cdots P_{2m_2,2m_2-k} \cdots P_{2m_2,0} \right) \cdots \\
&= \left( P_{2m_2,0} \cdots P_{2m_2,4} \cdots P_{2m_2,2m_2-j} \right) \\
&\cdots \left( P_{2m_2,k} \cdots P_{2m_2,2m_2-k} \cdots P_{2m_2,0} \right)
\end{align*}
\]

(28)

Then, the following theorem holds.

**Theorem 2.4.** For any \( m_2 \geq 2 \),

\[
\hat{X}_{2 \times m_2 \times 2} = \hat{P}_{2 \times m_2 \times 2} X_{2 \times m_2 \times 2} \hat{P}_{2 \times m_2 \times 2}.
\]

(30)

**Proof.** From Eq. (24), in \( Z_{2 \times m_2 \times 2} \) the position \((o_2, o_3)\) is the \( o \) in Eq. (16), where \( \alpha = 2(o_2 - 1) + o_3 \). Define

\[
\ell_o \equiv 1 + 2u_{2 \alpha_3} + u_{2 \alpha_1}.
\]

(31)

where \( \alpha \leq 4 \) and \( 1 \leq \alpha \leq 2m_2 \). For \( U = (u_{2 \alpha_3}, u_{2 \alpha_1}) \in \Sigma_{2 \times m_2 \times 2} \), from Theorem 2.1 it can be denoted by \( y_{2 \times m_2 \times 2, j_1, j_2, \ldots, j_{m_2}} \) and by Eq. (12) for fixed \( 1 \leq o_2 \leq m_2 \):

\[
J_{m_2} = 1 + 2u_{2 \alpha_1} + 2u_{2 \alpha_2} + 2u_{2 \alpha_3} + u_{2 \alpha_2},
\]

where \( 1 \leq J_{m_2} \leq 16 \). Accordingly, \( y_{0 \times J_{m_2}} \) can be represented by \( y_{2 \times m_2 \times 2, \ell_0} \) and the relation is

\[
\begin{align*}
y_{91} & \equiv y_{10} \oplus y_{21} \oplus \cdots \oplus y_{91} \\
y_{95} & \equiv y_{14} \oplus y_{25} \oplus y_{36} \oplus y_{47} \\
y_{109} & \equiv y_{1} \oplus y_{12} \oplus y_{23} \oplus y_{34} \\
y_{123} & \equiv y_{14} \oplus y_{15} \oplus y_{16} \oplus y_{17}
\end{align*}
\]

Therefore, from Eq. (19) in patterns ordering matrix \( \overline{X}_{2 \times m_2 \times 2} \) can be specified by

\[
y_{2 \times m_2 \times 2, j_1, j_2, \ldots, j_{m_2}} = y_{\ell_o} \oplus y_{\ell_o+1} \oplus \cdots \oplus y_{\ell_o+m_2} \\
= y_{\ell_o} \oplus y_{\ell_o+1} \oplus \cdots \oplus y_{\ell_o+2m_2-1} \oplus y_{\ell_o+2m_2-2}
\]

For any \( 1 \leq k \leq 2m_2 - 1 \),

\[
\begin{align*}
P_{2m_2,k} \hat{X}_{2 \times m_2 \times 2} &= P_{2m_2,k} \left( y_{\ell_o-k} \oplus y_{\ell_o-k+1} \oplus \cdots \oplus y_{\ell_o-2m_2} \right) \hat{P}_{2 \times m_2 \times 2} \\
&\equiv y_{\ell_o-2k} \oplus y_{\ell_o-2k+1} \oplus \cdots \oplus y_{\ell_o-2m_2}
\end{align*}
\]

is easily verified, such that \( P_{2m_2,k} \) exchanges \( \ell_o \) and \( \ell_o+1 \) in \( \hat{X}_{2 \times m_2 \times 2} \). Therefore, Eq. (30) follows from Eq. (29) and Lemma 2.3.

Now, according to Theorem 2.4,

\[
\hat{X}_{2 \times m_2 \times 2} = \left[ \hat{P}_{2 \times m_2 \times 2} \right]^{(t)}.
\]

(33)

for \( \Sigma_{2 \times m_2 \times 2} \) enables \( \hat{X}_{2 \times m_2 \times m_2} \) to be constructed for \( \Sigma_{2 \times m_2 \times m_3} \). Indeed, for fixed \( m_2 \geq 2 \) and \( m_3 \geq 2 \), let

\[
\hat{X}_{2 \times m_2 \times m_3, j_1, j_2, \ldots, j_{m_3}} = \left[ \hat{X}_{2 \times m_2 \times m_3, j_1, j_2, \ldots, j_{m_3}} \right]^{(t)}
\]

(32)

Therefore, by a similar argument as was used to establish Theorem 2.1 the following theorem holds for \( \Sigma_{2 \times m_2 \times m_3} \), the detailed proofs are omitted for brevity.

**Theorem 2.5.** For fixed \( m_2 \geq 2 \) and for any \( m_3 \geq 2 \), the ordering matrix \( \hat{X}_{2 \times m_2 \times m_3} \), with respect to \( \left[ \hat{P} \right] \) ordering can be expressed as

\[
\hat{X}_{2 \times m_2 \times m_3, k_1, k_2, \ldots, k_{m_3}} = \left[ \hat{X}_{2 \times m_2 \times m_3, k_1, k_2, \ldots, k_{m_3}} \right]^{(t)}
\]

where \( 1 \leq k_1 \leq 2m_2 \). For fixed \( 1 \leq k_1, k_2, \ldots, k_{m_3} \leq 2m_2 \), \( \hat{X}_{2 \times m_2 \times m_3, k_1, k_2, \ldots, k_{m_3-1}} = \left[ \hat{X}_{2 \times m_2 \times m_3, k_1, k_2, \ldots, k_{m_3-1}} \right]^{(t)} \), where \( \Sigma_{2 \times m_2 \times m_3, k_1, k_2, \ldots, k_{m_3-1}} \), is given by Eq. (32).

**Remark 2.6.** Similarly, according to other orderings, the following relations can be derived

\[
\begin{align*}
\hat{X}_{2 \times m_2 \times m_3} &= \left[ \hat{X}_{2 \times m_2 \times m_3} \right]^{(t)} \oplus \hat{X}_{2 \times m_2 \times m_3} \oplus \cdots \oplus \hat{X}_{2 \times m_2 \times m_3} \oplus \hat{X}_{2 \times m_2 \times m_3} \\
\hat{Y}_{m_2 \times m_3} &= \left[ \hat{X}_{m_2 \times m_3} \right]^{(t)} \oplus \hat{X}_{m_2 \times m_3} \oplus \cdots \oplus \hat{X}_{m_2 \times m_3} \\
\hat{Y}_{m_3 \times m_2} &= \left[ \hat{X}_{m_3 \times m_2} \right]^{(t)} \oplus \hat{X}_{m_3 \times m_2} \oplus \cdots \oplus \hat{X}_{m_3 \times m_2}
\end{align*}
\]
3. Transition Matrices and Spatial Entropy

3.1. Transition matrices

Based on the definitions of the ordering matrices \(X_{2\times m_2 \times m_3}\) for \(\Sigma_{2\times m_2 \times m_3}\) having been defined, high order transition matrices \(A_{2\times 2\times 2\times 2}\) can now be derived from \(A_{2\times 2\times 2}\). As in the two-dimensional case [Ban & Lin, 2006], a basic set \(B \subset \Sigma_{2\times 2}\) is assumed to be given. Define the transition matrix 

\[ A_{2\times 2\times 2\times 2}(B) = \begin{bmatrix} a_{i,j,k,l} \end{bmatrix} \]

where \(a_{i,j,k,l} = 1\) if \(x_i x_j \in B\), \(= 0\) otherwise. (33)

Then, the transition matrix \(A_{2\times 2\times 2\times 2}\) is a \(2^{m_2} \times 2^{m_3} \times 2^{m_4}\) matrix with entries \(a_{i,j,k,l} = a_{i,j,k,l} \in B\), \(= 0\) otherwise. (34)

\[ a_{i,j,k,l} = \prod_{k=1}^{m_2} a_{i,j,k,l} \prod_{k=1}^{m_3} a_{i,j,k,l} \prod_{k=1}^{m_4} a_{i,j,k,l} \]  

(35)

Before \(A_{2\times 2\times 2\times 2}\) is introduced, three products of the matrices are defined as follows.

**Definition 3.1.** For any two matrices \(M = (M_{ij})\) and \(N = (N_{ij})\), the Kronecker product (tensor product) \(M \otimes N\) of \(M\) and \(N\) is defined by

\[ M \otimes N = (M_{ij} \otimes N_{ij}) \]

For any \(n \geq 1\),

\[ \otimes^nm = M \otimes N \otimes \cdots \otimes N, \]

\(n\)-times in \(N\).

Next, for any two \(m \times m\) matrices \(P = (P_{ij})\) and \(Q = (Q_{ij})\)

where \(P_{ij}\) and \(Q_{ij}\) are numbers or matrices, the Hadamard product \(P \circ Q\) is defined by

\[ P \circ Q = (P_{ij} \cdot Q_{ij}) \]

where the product \(P_{ij} \cdot Q_{ij}\) of \(P_{ij}\) and \(Q_{ij}\) may be a multiplication of numbers, of numbers and matrices or of matrices whenever it is well-defined.

Finally, product \(\oplus\) is defined as follows. For any \(4 \times 4\) matrix

\[ M_2 = \begin{bmatrix} m_{11} & m_{12} & m_{21} & m_{22} \\ m_{13} & m_{14} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{41} & m_{42} \\ m_{33} & m_{34} & m_{43} & m_{44} \end{bmatrix} = \begin{bmatrix} M_{2,1} & M_{2,2} \\ M_{2,3} & M_{2,4} \end{bmatrix} \]

and any \(2 \times 2\) matrix

\[ N = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \]

where \(m_{ij}\) are numbers and \(N_k\) are numbers or matrices, for \(1 \leq i, j, k \leq 4\), define

\[ M_2 \oplus N = \begin{bmatrix} m_{11}N_1 & m_{12}N_2 & m_{21}N_1 & m_{22}N_2 \\ m_{13}N_3 & m_{14}N_4 & m_{23}N_3 & m_{24}N_4 \\ m_{31}N_1 & m_{32}N_2 & m_{41}N_1 & m_{42}N_2 \\ m_{33}N_3 & m_{34}N_4 & m_{43}N_3 & m_{44}N_4 \end{bmatrix} \]

Furthermore, for \(n \geq 1\), the \(n + 1\)-th order of transition matrix of \(M_2\) is defined by

\[ M_{n+1} = M_2 \oplus (M_2^{-1}) \]

\[ \begin{bmatrix} M_{n+1,1} & M_{n+1,2} \\ M_{n+1,3} & M_{n+1,4} \end{bmatrix} = \begin{bmatrix} m_{11}M_{n,1} & m_{12}M_{n,2} & m_{21}M_{n,1} & m_{22}M_{n,2} \\ m_{13}M_{n,3} & m_{14}M_{n,4} & m_{23}M_{n,3} & m_{24}M_{n,4} \\ m_{31}M_{n,1} & m_{32}M_{n,2} & m_{41}M_{n,1} & m_{42}M_{n,2} \\ m_{33}M_{n,3} & m_{34}M_{n,4} & m_{43}M_{n,3} & m_{44}M_{n,4} \end{bmatrix} \]

\[ M_{n+1} = M_2 \oplus (M_2^{-1}) \]

\[ \begin{bmatrix} M_{n+1,1} & M_{n+1,2} \\ M_{n+1,3} & M_{n+1,4} \end{bmatrix} \]

where

\[ M_n = \otimes^m N = \begin{bmatrix} M_{n,1} & M_{n,2} \\ M_{n,3} & M_{n,4} \end{bmatrix} \]

Here, the following convention is adopted,

\[ \otimes^0M = E_2 \]

where \(E_2\) is the \(2 \times 2\) matrix with 1 as its entries.

Theorem 2.1, yields results for \(A_{2\times 2\times 2\times 2}\) as \(T_n\) in Theorem 3.1 in [Ban & Lin, 2006]. Indeed,
Theorem 3.2. Let $A_{2×2×2}$ be a transition matrix that is given by Eqs. (33) and (34). Then, for high order transition matrices $A_{2×m_2×2}$, $m_2 ≥ 3$, the following three equivalent statements hold:

(I) $A_{2×m_2×2}$ can be decomposed into $m_2$ successive $4 × 4$ matrices

$$A_{2×m_2×2} = [A_{2×2×2}j_{i}]_{4×4},$$

where $1 ≤ j_i ≤ 16$. For fixed $1 ≤ j_1, j_2, ..., j_{m_2} ≤ 16$, $A_{2×m_2×2}j_{1}j_{2}j_{3}j_{4} = [A_{2×2×2}j_{1}j_{2}j_{3}j_{4}]_{4×4},$ where $1 ≤ j_{k+1} ≤ 16$ and $1 ≤ k ≤ m_2 − 1$. For fixed $j_1, j_2, ..., j_{m_2−1} ∈ \{1, 2, ..., 16\}$, $A_{2×m_2×2}j_{1}j_{2}j_{3}j_{4} = [a_{2×2×2}j_{1}j_{2}j_{3}j_{4}]_{4×4},$ where $a_{2×2×2}j_{1}j_{2}j_{3}j_{4}$ is defined in Eq. (35).

(II) Starting from

$$A_{2×2×2} = [A_{2×2×2}j_{i}]_{4×4}$$

and

$$A_{2×2×2}j_{i}j_{j}j_{k}j_{l} = [a_{2×2×2}j_{i}j_{j}j_{k}j_{l}]_{4×4},$$

for $m_2 ≥ 3$, $A_{2×m_2×2}$ can be obtained from $A_{2×2×2}$ by replacing $A_{2×2×2}j_{i}j_{j}j_{k}j_{l}$ with

$$(A_{2×2×2}j_{i}j_{j}j_{k}j_{l})_{4×4} = (A_{2×2×2}j_{i}j_{j}j_{k}j_{l})_{4×4},$$

(III) For $m_2 ≥ 3$,

$$A_{2×m_2×2} = (A_{2×m_2−1×2}j_{1}j_{2}j_{3}j_{4} (E_{2m_2−2} ⊗ A_{2×2×2}),$$

where $E_{2^m}$ is the $2^m × 2^m$ matrix with 1 as its entries.

Proof

(I) The proof involves simply replacing $A_{2×m_2×2}$ by $a_{2×m_2×2}$ and $A_{2×m_2×2}$ in Theorem 2.1, respectively.

(II) follows directly from (I).

(III) follows from (I): $A_{2×m_2×2} = [A_{2×m_2×2}j_{i}j_{i}], 1 ≤ j_i ≤ 2^4$, (I) yields the following formula;

$$A_{2×m_2×2} = [a_{2×2×2}j_{i}j_{i}j_{i}j_{i}]_{4×4},$$

where $1 ≤ j_{k+1} ≤ m_2 − 2$. For fixed $1 ≤ k_1, k_2, ..., k_{m_2−1} ≤ 2^4$, $A_{2×m_2×2}j_{k_1}j_{k_2}j_{k_3}j_{k_4} = [a_{2×m_2×2}j_{k_1}j_{k_2}j_{k_3}j_{k_4}]_{4×4},$ where $1 ≤ k_{m_2−1} ≤ 2^m$ and by Eq. (37)

$$a_{2×m_2×2}j_{k_1}j_{k_2}j_{k_3}j_{k_4} = \prod_{\ell=1}^{m_2−1} a_{2×m_2×2}j_{\ell}k_{\ell+1}.$$

Remark 3.3. As stated in Remark 2.2, the following formulae apply

$$A_{2×2×m_2} = [a_{2×2×m_2}j_{k_1}j_{k_2}j_{k_3}j_{k_4}]_{2^{m_2}×2^{m_2}},$$

$$A_{2×m_2×2} = [a_{2×m_2×2}j_{k_1}j_{k_2}j_{k_3}j_{k_4}]_{2^{m_2}×2^{m_2}},$$

$$A_{2×m_2×2} = [a_{2×m_2×2}j_{k_1}j_{k_2}j_{k_3}j_{k_4}]_{2^{m_2}×2^{m_2}},$$

Theorem 3.4

$$A_{2×m_2×2} = P_{2×m_2×2}A_{2×m_2×2}P_{2×m_2×2}.$$
For any \( m_3 \geq 3 \), \( A_{2;2,m_3} \) can be obtained from \( A_{2;2,m_3,2(m_3-1)} \) by replacing
\[
A_{2;2,m_3} x_2 k_1 \quad \text{with} \quad (A_{2;2,m_3} x_2 k_1)^{m_3} \circ (A_{2;2,m_3} x_2)^{m_3-2}.
\]

Furthermore, for \( m_3 \geq 3 \),
\[
A_{2;2,m_3} = (A_{2;2,m_3,(m_3-1)}^{m_3} \circ (A_{2;2,m_3} x_2)^{m_3-2}) \circ (E_{m_3} \otimes A_{2;2,m_3}).
\]

The proof closely resembles that of Theorems 2.1 and 2.2. Details of the proof are obvious and repeated, hence can be omitted.

Remark 3.6. As in Remark 2.6, the following formulae are obtained
\[
\begin{align*}
& A_{2;2,m_3} = [g_{2;2,m_3,m_3,1(2,m_3-1)}^{m_3} x_2^{m_3} x_2] \\
& A_{2;2,m_3} = [g_{2;2,m_3,m_3,1(2,m_3-1)}^{m_3} x_2^{m_3} x_2] \\
& A_{2;2,m_3} = [g_{2;2,m_3,m_3,1(2,m_3-1)}^{m_3} x_2^{m_3} x_2] \\
& A_{2;2,m_3} = [g_{2;2,m_3,m_3,1(2,m_3-1)}^{m_3} x_2^{m_3} x_2] \\
& A_{2;2,m_3} = [g_{2;2,m_3,m_3,1(2,m_3-1)}^{m_3} x_2^{m_3} x_2].
\end{align*}
\]

Finally, the spatial entropy \( h(B) \) can be computed from the maximum eigenvalue \( \lambda_{2;2,m_2,m_3} \) of \( A_{2;2,m_2,m_3} \). Indeed,

Theorem 3.7. Let \( \lambda_{2;2,m_2,m_3} \) be the maximum eigenvalue of \( A_{2;2,m_2,m_3} \), then
\[
h(B) = \lim_{m_2,m_3 \to \infty} \frac{\log \lambda_{2;2,m_2,m_3}}{m_2 m_3}.
\]

Proof. By the same arguments as in [Chow et al., 1996a], the limit Eq. (1) is well-defined and exists. From \( A_{2;2,m_2,m_3} \), for \( m_2 \geq 2 \) and \( m_3 \geq 2 \),
\[
\Gamma_{2;2,m_2,m_3}(B) = \sum_{1 \leq j < 2^{m_3}} (A_{2;2,m_2,m_3}^{m_2})_{ij} = \left( (A_{2;2,m_2,m_3}^{m_2})_{11} \right).
\]

As in the one-dimensional case,
\[
\lim_{m_2,m_3 \to \infty} \frac{\log \left( (A_{2;2,m_2,m_3}^{m_2})_{11} \right)}{m_2 m_3} = \log \lambda_{2;2,m_2,m_3},
\]
as for example [Ban & Lin, 2005]. Hence,
\[
h(B) = \lim_{m_2,m_3 \to \infty} \frac{\log \Gamma_{2;2,m_2,m_3}(B)}{m_2 m_3} = \lim_{m_2,m_3 \to \infty} \frac{1}{m_2 m_3} \lim_{m_2 \to \infty} \frac{\log \Gamma_{2;2,m_2,m_3}(B)}{m_2 m_3}.
\]

The detailed proofs are as above.

3.2. Computation of \( \lambda_{m,n} \) and entropies

The last subsection provided a systematic means of writing down \( A_{2;2,m_2,m_3} \) from \( A_{2;2,x} \). As in a two-dimensional case [Ban & Lin, 2005], a recursive formula for \( A_{2;2,m_2,m_3} \) can be obtained in a special structure. An illustrative example is presented in which \( A_{2;2,m_2,m_3} \) can be derived explicitly to demonstrate the methods developed in the preceding subsection. More complete results will be presented later.

Let
\[
G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad E = E_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]
and
\[
A_{2;2} = G \otimes E \otimes E \otimes E.
\]

Proposition 3.9. Substitute \( A_{2;2} \) into Eqs. (41) and (42). Then,
\[
(i) \quad A_{2;2} x_2 = \cdots (G \otimes E) \otimes (E^2),
(ii) \quad A_{2;2} x_2 = \cdots (G^3) \otimes (E^2),
(iii) \quad A_{2;2} x_2 = \cdots (G^4) \otimes (E^2).
\]
Furthermore, for the maximum eigenvalue $\lambda_{2, m_2, m_3}$ of $A_{2, m_2, m_3}$, the following recursive formula holds:

$$\lambda_{2, m_2, m_1+1, m_3} = 2g^{m_1-1}\lambda_{2, m_2, m_1, m_3}$$

and

$$\lambda_{2, m_2, m_1+1} = 2g^{m_1-1}\lambda_{2, m_2, m_1, m_3}$$

for $m_2, m_3 \geq 2$ with

$$\lambda_{2, 2, 2} = 2^g.$$ (48)

The spatial entropy is

$$h(A_g; x; y; z) = \log g,$$ (49)

where $g = (1 + \sqrt{5})/2$, the golden-mean.

Theorem 3.1, then matrices $A_{g, 2, 2, 2}$ where

$$A_{g, 2, 3, 2} = (A_{g, 2, 2, 2})_{4x4} \otimes (E_{2} \otimes A_{g, 2, 2, 2})_{4x4}$$

$$= (G \otimes E \otimes G \otimes E)_{4x4} \otimes (E \otimes E \otimes (G \otimes E \otimes E \otimes E))_{4x4}$$

$$= (G \otimes E) \otimes (E \otimes G) \otimes (E \otimes E \otimes (E \otimes E \otimes E))_{4x2}$$

Assume that $A_{g, 2, x(m_1-1), 2} = (G \otimes E)^{m_2} \otimes (\otimes E^2)$. Then by Eq. (36) again,

$$A_{g, 2, x(m_1), 2} = (A_{g, 2, x(m_1-1), 2}) \odot ((G \otimes E)^{m_2-2} \otimes (\otimes E^2))$$

$$= (G \otimes E)_{2x2} \otimes (G \otimes E)_{2x2} \otimes (G \otimes E)_{2x2} \otimes (G \otimes E)_{2x2}$$

The following property of matrices is required and detailed proofs are omitted: For any two $2 \times 2$ matrices $A$ and $B$, $P = (A \otimes B)B = B \otimes A$,

where $P$ is given by Eq. (28). Equation (44) is proven by induction on $m_2$. When $m_2 = 2$, by Theorem 3.1,

$$A_{2, x, x, 2} = P_{2, x, x, 2}A_{2, x, x, 2} = (P_{g, 2}A_{g, 2, x, 2})_{2x2}$$

by Eq. (50).

Now, Eq. (44) is assumed to hold for $m_2 - 1$;

$$A_{g, 2, x(m_1-1), x} = (G^{m_1}) \otimes (\otimes E^{m_1}).$$

Then

$$A_{g, 2, x(m_1), x} = P_{2, x, x, 2}A_{2, x, x, 2}P_{2, x, x, 2}$$

$$= (P_{m_2, 2}P_{m_2, 2} \cdots P_{m_2, 2}) \cdot (P_{m_2, 2}P_{m_2, 2} \cdots P_{m_2, 2}) \cdots (P_{m_2, 2})$$

$$= (P_{2, m_2, 2}P_{2, m_2, 2} \cdots P_{2, m_2, 2}) \cdot (P_{2, m_2, 2}P_{2, m_2, 2} \cdots P_{2, m_2, 2}) \cdots (P_{2, m_2, 2})$$
m and Theorem 3.7. The proof is thus complete.

Similarly, for a fixed $m$, Eq. (40) is easily verified. Equation (41) is established for fixed $m_3$ using Eq. (45), yielding

\[
\lambda_{1,2,m_2,m_3} = (G \otimes (G^{m_2-1} \otimes (G^{m_3} \otimes E))^{m_3-2} \otimes (E^{m_2+1})),
\]

which implies

\[
\lambda_{1,2,m_2,m_3} = 2^{m_3} - 1 \lambda_{1,2,m_2,m_3},
\]

see [Bellman, 1970; Gantmacher, 1959] and [Horn & Johnson, 1990].

Similarly, for a fixed $m_2$, Eq. (47) is proven using Eq. (45) again:

\[
\lambda_{1,2,m_2,m_3} = (G \otimes (G^{m_2-1} \otimes E)^{m_2} \otimes (E^{m_3+1}))\]

which implies

\[
\lambda_{1,2,m_2,m_3} = 2^{m_3} - 1 \lambda_{1,2,m_2,m_3},
\]

Finally, Eq. (49) follows from Eqs. (46) and (47) and Theorem 3.7. The proof is thus complete.

### 4. Connecting Operator

This section introduces the connecting operator and employs it to derive a recursive formula between an elementary pattern of order $(m_1, m_2, m_3)$ and that of order $(m_1, m_2, m_3)$. It is also adopted to obtain a lower bound on entropy.

#### 4.1. Connecting operator in z-direction

This subsection derives connecting operators and studies their properties. For brevity, only the connecting operator in the $z$-direction is discussed but the other cases are similar, and will be considered in the following remarks. For clarity, as in the former section, two symbols on lattice $Z^2 \times Z^2$ are examined first.

According to Theorem 3.5, the transition matrix $A_{2,2,m_2,m_3}$ can be represented as $A_{2,2,m_2,m_3}$, where $1 \leq m_3 \leq 2m_2$, is a $2m_2(m_2-1) \times 2m_2(m_2-1)$ matrix.
For matrix multiplication, the indices of \( A_{i,j,k}^{m_2,m_3,x} \) are conveniently expressed as

\[
\begin{bmatrix}
A_{i,j,k}^{m_2,m_3,x}_{11} & A_{i,j,k}^{m_2,m_3,x}_{12} & \cdots & A_{i,j,k}^{m_2,m_3,x}_{1,m_2} \\
A_{i,j,k}^{m_2,m_3,x}_{21} & A_{i,j,k}^{m_2,m_3,x}_{22} & \cdots & A_{i,j,k}^{m_2,m_3,x}_{2,m_2} \\
\vdots & \vdots & \ddots & \vdots \\
A_{i,j,k}^{m_2,m_3,x}_{m_2,1} & A_{i,j,k}^{m_2,m_3,x}_{m_2,2} & \cdots & A_{i,j,k}^{m_2,m_3,x}_{m_2,m_2}
\end{bmatrix}
\]

Clearly, \( A_{i,j,k}^{m_2,m_3,x} = A_{i,j,m_3,k_1,m_2,k_2} \), where \( \alpha = \alpha(j_1,j_2) = 2^{m_2}(j_2 - 1) + j_1 \). For \( m_1 \geq 2 \), the elementary pattern in the entries of \( A_{i,j,k}^{m_1,m_2,m_3,x} \) is given by

\[
A_{i,j,k}^{m_1,m_2,m_3,x} = A_{i,j,m_3,k_1,m_2,k_2}
\]

where \( \beta \in \{1, 2, \ldots, 2^{m_2}\} \) and \( 1 \leq r \leq m_1 + 1 \). A lexicographic order for multiple indices \( I_{m_1+1} \) is introduced, using

\[
K(I_{m_1+1}) = 1 + \sum_{r=2}^{m_1} 2^{m_2}(m_1 - r)(j_r - 1).
\]

Now, \( A_{i,j,k}^{m_2,m_3,x} \) can be represented by

\[
A_{i,j,k}^{m_2,m_3,x} = A_{i,j,m_3,k_1,m_2,k_2}A_{i,j,m_3,k_1,m_2,k_3} \cdots A_{i,j,m_3,k_1,m_2,k_{m_2+1}},
\]

where

\[
\alpha = \alpha(j_1, j_2, j_3, \ldots, j_{m_2}) = 2^{m_2}(j_1 - 1) + \beta_{m_2+1}
\]

and

\[
k = K(I_{m_1+1})
\]

as in Eq. (51). Accordingly, \( A_{i,j,k}^{m_2,m_3,x} \) can be expressed as

\[
[A_{i,j,m_3,k_1,m_2,k_2}]^{m_2 \times m_2},
\]

where \( 1 \leq \alpha \leq 2^{m_2} \), and

\[
A_{i,j,k}^{m_2,m_3,x} = \sum_{k=1}^{2^{m_2}(m_2+1)} A_{i,j,k}^{(k)}
\]

Moreover,

\[
V_{i,m_2,m_3,x} = (A_{i,j,m_3,k_1,m_2,k_2})',
\]

where \( 1 \leq k \leq 2^{m_2}(m_2+1) \). \( V_{i,m_2,m_3,x} \) is a \( 2^{m_2}(m_2+1) \) column vector that comprises all elementary patterns in \( A_{i,j,m_2,m_3,x} \). The ordering matrix \( V_{i,m_2,m_3,x} \) is now defined as

\[
[V_{i,m_2,m_3,x}]^{m_2 \times m_2},
\]

where \( 1 \leq \alpha \leq 2^{m_2} \). The ordering matrix \( V_{i,m_2,m_3,x} \) allows the elementary patterns to be tracked during the reduction from \( A_{i,j,k}^{m_2,m_3,x} \). This careful book-keeping constitutes a systematic way to generate the admissible patterns, and as in Sec. 4.2, lower-bound estimates of spatial entropy.

This simplest example is considered first to illustrate this concept.

Example 4.1. For \( m_1 = 2, m_2 = 3, m_3 = 3 \), the following can be easily verified:

\[
A_{i,j,k}^{2,3,3,x} = [A_{i,j,k}^{2,3,3,x}]^{2^2 \times 2^3},
\]

where \( 1 \leq \alpha \leq 2^6 \) and

\[
A_{i,j,k}^{2,3,3,x} = \sum_{k=1}^{2^6} A_{i,j,k}^{(k)},
\]

and for fixed \( \alpha_1 \) and \( k \) the represented pattern of \( A_{i,j,k}^{(k)} \), in the following form.

\[
\begin{array}{ccc}
\times & \times & \times \\
\times & \times & \times \\
\times & \times & \times \\
\end{array}
\]

If the red symbol is defined equal to 1, and white symbol equals 0, then \( \alpha_1 = 2^{\alpha_1_1} + 2^{\alpha_1_2} + 2^{\alpha_1_3} + 2^{\alpha_1_4} + 2^{\alpha_1_5} + \alpha_1_6 + 1 \) and \( k = 2^6k_1 + 2k_2 + k_3 + 1 \). Hence

\[
V_{i,j,k}^{2,3,3,x} = (A_{i,j,k}^{(k)})',
\]

where \( 1 \leq k \leq 2^6 \) and \( 1 \leq \alpha_1 \leq 2^6 \). Define

\[
V_{i,j,k}^{2,3,3,x} = (A_{i,j,k}^{(k)})',
\]
where $1 \leq k \leq 2^3$ and $1 \leq \alpha_1, \alpha_2 \leq 2^6$ and the represented pattern of $A_{2,3,3,\alpha_1,\alpha_2}$ is

Therefore, for instance,

$$V_{3;2,3,1,1} = S_{i,m_3;2,3,1} V_{3;2,3,1},$$

and the represented patterns of $S_{i,m_3;2,3,1}$
The above derivation reveals that $V_{2\times 3,3\times m_2}$ can be reduced to $V_{3,3\times m_2}$ by multiplication using connecting operator $S_{2,3\times m_2}$. This procedure can be extended to introduce the connecting operator $S_{3\times m_1,m_2} = [S_{E,m_1,m_2}]_{i,j}^2$, where $1 \leq \alpha_1, \alpha_2 \leq 2^{m_1}$, for all $m_1 \geq 2$, $m_2 \geq 2$.

**Definition 4.2.** For $m_1 \geq 2$, $m_2 \geq 2$, define

$$C_{E,m_1,m_2,i,j}^2 \equiv [A_{E,2\times m_1,2\times m_2}]_{i,j}^2 \circ [A_{E,2\times (m_1-1),2\times m_2}]_{i,j}^2 \circ [E_{C,2\times m_1},2\times m_2]_{i,j}^2 \circ \{C_{E,m_1,m_2,i,j}^2 \circ \{C_{E,m_1,m_2,i,j}^2 \circ [E_{C,2\times m_1},2\times m_2]\}_{i,j}^2\}_{i,j}^2$$

where the row matrix $S_{E,m_1,m_2}$ of $S_{E,m_1,m_2}$ is defined in Eqs. (13) and (14). And

$$C_{E,m_1,m_2,i,j}^2 \equiv [A_{E,2\times m_1,2\times m_2}]_{i,j}^2 \circ [A_{E,2\times (m_1-1),2\times m_2}]_{i,j}^2 \circ [E_{C,2\times m_1},2\times m_2]_{i,j}^2 \circ \{C_{E,m_1,m_2,i,j}^2 \circ \{C_{E,m_1,m_2,i,j}^2 \circ [E_{C,2\times m_1},2\times m_2]\}_{i,j}^2\}_{i,j}^2$$

where $(A_{E,2\times m_1,2\times m_2}^{(r)})_{i,j}^2$ is the $i,j$th block of the matrix $(A_{E,2\times m_1,2\times m_2}^{(r)})$, $(A_{E,2\times m_1,2\times m_2}^{(r)})_{i,j}^2$ is the column matrix of $(A_{E,2\times m_1,2\times m_2}^{(r)})_{i,j}^2$ and $E_k$ is the $2^k \times 2^k$ matrix with 1 as its entries.

**Remark 4.3.** By a similar method, the following connecting operators can also be defined.

**Theorem 4.4.** For any $m_2 \geq 2$, $m_1 \geq 2$ and $1 \leq i_1, i_2 \leq 2^{m_2}$,

$$C_{E,m_1,m_2,i_1,i_2}^2 = [a_{E,2\times m_1,2\times m_2}]_{i_1,i_2}^2 \circ C_{E,m_1,m_2,i_1,i_2}^2,$$

where $1 \leq i \leq 2^{m_2}$.

**Proof.** By Theorem 3.5 and Remark 3.6,

$$A_{E,2\times m_1,2\times m_2} = [a_{E,2\times m_1,2\times m_2}]_{i_1,i_2}^2 \circ A_{E,2\times (m_1-1),2\times m_2},$$

where $1 \leq i_1 \leq 2^{m_2}$. Hence, by

$$C_{E,m_1,m_2,i_1,i_2}^2 = [a_{E,2\times m_1,2\times m_2}]_{i_1,i_2}^2 \circ C_{E,m_1,m_2,i_1,i_2}^2 \circ \{C_{E,m_1,m_2,i_1,i_2}^2 \circ \{C_{E,m_1,m_2,i_1,i_2}^2 \circ [E_{C,2\times m_1},2\times m_2]\}_{i_1,i_2}^2\}_{i_1,i_2}^2$$

where $1 \leq i_1 \leq 2^{m_2}$. The proof is complete.

Notably, Eq. (59) implies $C_{E,m_1,m_2,i,j}$ is

$$[a_{E,2\times m_1,2\times m_2}]_{i,j}^2 \circ C_{E,m_1,m_2,i,j}^2 \circ \{C_{E,m_1,m_2,i,j}^2 \circ \{C_{E,m_1,m_2,i,j}^2 \circ [E_{C,2\times m_1},2\times m_2]\}_{i,j}^2\}_{i,j}^2$$

where $1 \leq i \leq 2^{m_2}$.
with \( i = x \) and \( m_x+1 = j \). \( C_{x,m_x,1,m_{x+1}} \) comprises all paths of length \( m_x + 1 \), that start at \( x \) and end at \( j \). Indeed, the entries of \( C_{x,m_x,1,m_{x+1}} \) and \( A_{x,m_x+1,m_{x+1}} \) are the same. However, the arrangements differ.

Substituting \( m_x \) for \( m_x + 1 \) into Eq. (32) and using Eq. (38), \( A_{y,m_x,m_{x+1}}^{(k)} \) could be represented by

\[
A_{x,m_x+1,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\beta_{x,m_x+1}) = \prod_{\ell = 1}^{m_x+1}(A_{x,m_x+1,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\beta_{x,m_x+1}))
\]

where \( 1 \leq \hat{A}_{1}, \hat{A}_{2} \leq 2^{m_x} \) and \( \alpha_{x} = \alpha(\hat{A}_{1}, \hat{A}_{2}, x) \) and \( \hat{\alpha} = \hat{\alpha}(\hat{A}_{1}, \hat{A}_{2}) \) for \( 1 \leq x \leq m_x + 1 \).

After \( m_x \) matrix multiplications have been performed as in Eq. (60),

\[
A_{x,m_x,m_{x+1}}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell) = [A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell)]^{2^{m_x} \times 2^{m_x}}
\]

where \( 1 \leq \alpha_{x} \leq 2^{m_x} \) and \( A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell) \) can be represented by

\[
\sum_{\ell = 1}^{2^{m_x}(m_x-1)} K(\hat{A}_{x,m_x+1,m_{x+1}+1}, \alpha_{x}, k, \ell) A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell)
\]

which is a linear combination of \( A_{x,m_x+1,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell) \) with the coefficients \( K(\hat{A}_{x,m_x+1,m_{x+1}+1}, \alpha_{x}, k, \ell) \) which are products of \( a_{x,2^{m_x} \times 2^{m_x}} \). \( 1 \leq j \leq m_x \), \( K(\hat{A}_{x,m_x+1,m_{x+1}+1}, \alpha_{x}, k, \ell) \) must be studied in more detail. Notably,

\[
A_{x,2^{m_x} \times 2^{m_x}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell) = [A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell)]^{2^{m_x} \times 2^{m_x}}
\]

where \( 1 \leq \alpha_{x} \leq 2^{m_x} \),

\[
A_{x,m_x+1,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell) = \sum_{k = 1}^{2^{m_x}(m_x-1)} A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell)
\]

and

\[
A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell) = [\sum_{k = 1}^{2^{m_x}(m_x-1)} A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell)]^{2^{m_x} \times 2^{m_x}}
\]

where \( 1 \leq \alpha_{x} \leq 2^{m_x} \). Now, \( V_{y,m_x,m_{x+1} \times 3} \) is defined as

\[
V_{y,m_x,m_{x+1} \times 3} = (A_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell))^{2^{m_x} \times 2^{m_x}}
\]

From Eqs. (62) and (64),

\[
K(\hat{A}_{x,m_x+1,m_{x+1}+1}, \alpha_{x}, k, \ell) = K(\hat{A}_{x,m_x+1,m_{x+1}+1}, \alpha_{x}, k, \ell)
\]

where \( 1 \leq \beta_{x,1} \leq 2^{m_x} \) and \( \hat{\beta} = \hat{\beta}(\hat{A}_{1}, \hat{A}_{2}) \) for \( 1 \leq x \leq m_x \), \( \beta_{x,1} \leq 2^{m_x}(m_x-1) \) is a \( 2^{m_x}(m_x-1) \times 2^{m_x}(m_x-1) \) matrix. Now

\[
K(\hat{A}_{x,m_x+1,m_{x+1}+1}, \alpha_{x}, k, \ell) = S_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell)
\]

must be shown as follows.

**Theorem 4.5.** For any \( m_x \geq 2 \), \( m_x \geq 2 \) and \( m_x \geq 3 \), let \( S_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell) \) be given as in Eqs. (57) and (58). Then,

\[
V_{x,m_x,m_{x+1} \times 3} = S_{x,m_x,m_{x+1}}^{(k)}(\hat{A}_{x,m_x,m_{x+1}},\alpha_{x}, k, \ell)
\]

or equivalently, the recursive formula.
Proof. From Eq. (61), \( A^{(k)}_{m_1, m_2, m_3+1; \alpha_1, \alpha_2} \) can be represented as the pattern:

\[
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_{p+1} \\
\ell_1 & \ell_2 & \cdots & \ell_{p+1} \\
\vdots & \vdots & \ddots & \vdots \\
\ell_{p+1} & \ell_{p+1} & \cdots & \ell_{p+1} \\
\end{array}
\]

From Definition 4.2, \( S^m_{m_1, m_2, m_3; \alpha_1, \alpha_2} \) represents the following pattern:

\[
\begin{array}{cccc}
\alpha_1 & \alpha_2 & \cdots & \alpha_{p+1} \\
\ell_1 & \ell_2 & \cdots & \ell_{p+1} \\
\vdots & \vdots & \ddots & \vdots \\
\ell_{p+1} & \ell_{p+1} & \cdots & \ell_{p+1} \\
\end{array}
\]

Therefore, Eq. (67) follows from Eqs. (69)–(71).

Next, Eq. (68) follows simply from Eqs. (69) and (71).

For any positive integer \( p \geq 2 \), applying Theorem 4.5 \( p \) times allows the elementary patterns of \( A^{m_1}_{m_2, m_3+1; \alpha_1, \alpha_2} \) to be expressed as products of a sequence of \( S^m_{m_1, m_2, m_3; \alpha_1, \alpha_2} \) and the elementary patterns in \( A^{m_1}_{m_2, m_3+1; \alpha_1, \alpha_2} \). The elementary pattern in \( A^{m_1}_{m_2, m_3+1; \alpha_1, \alpha_2} \) is first considered. For any \( p \geq 2 \) and \( 1 \leq q \leq p-1 \), define

\[ A^{(k)}_{m_1, m_2, m_3+1+q; \alpha_1, \alpha_2; \cdots; \alpha_q} = [A^{(k)}_{m_1, m_2, m_3+1+q; \alpha_1, \alpha_2; \cdots; \alpha_q}]^{2^{m_2} \times 2^{m_3}}, \]

where \( 1 \leq \alpha_{q+1} \leq 2^{m_2} \). Then

\[ A^{(k)}_{m_1, m_2, m_3+1+q; \alpha_1, \alpha_2; \cdots; \alpha_{p+1}} \]

can be represented as

\[
\sum_{k=1}^{2^{m_2}(m_3-1)} \sum_{k=1}^{2^{m_2}(m_3-1)} \cdots \sum_{k=1}^{2^{m_2}(m_3-1)}
\times \prod_{k=2}^{p+1} K(m_1, m_2; \alpha_{q+1}; \ell_{q+1}; k)
\times A^{(k)}_{m_1, m_2, m_3+1; \alpha_1, \alpha_2; \cdots; \alpha_q} \quad (72)
\]

where and \( \ell_1 = k \) can be easily verified.

Hence, for any \( p \geq 2 \), Eq. (65) can be generalized for \( A^{m_1}_{m_2, m_3; \alpha_1, \alpha_2} \) as a \((2^{m_2})^{p+1} \times (2^{m_3})^{p+1}\)
The proof is complete.

\[ A_{i,j}^{(m)} = A_{i,j}^{(m,2^{n-1})} \]

where

\[ A_{i,j}^{(m,2^{n-1})} = \sum_{k=1}^{2^{(m-1)-1}} A_{i,j}^{(k)} \]

In particular, if \( \alpha_1, \alpha_2, \ldots, \alpha_{p+1} \in \{ 2^{m_1}(s-1) + s | 1 \leq s \leq 2^{m_2} \} \) then \( A_{i,j}^{(m,2^{m_2})} \) lies on the diagonal of \( A_{i,j}^{(m)} \).

Therefore, Theorem 4.5 can be generalized to the following theorem.

**Theorem 4.6.** For any \( m_1 \geq 2, m_2 \geq 2, m_3 \geq 2 \) and \( p \geq 1 \), \( V_{i,j}^{(m_1,2^{m_2}+p,\alpha_1,\alpha_2,\ldots,\alpha_{p+1})} \) can be represented as

\[ S_{i,j}^{(m_1,2^{m_2}+p,\alpha_1,\alpha_2,\ldots,\alpha_{p+1})} \]

where \( 1 \leq \alpha_i \leq 2^{m_1} \) and \( 1 \leq i \leq p + 1 \).

**Proof.** From Eqs. (72), (65) and (67),

\[
A_{i,j}^{(k)} = \sum_{\ell_1=1}^{2^{m_1}(s-1) + s} \sum_{\ell_2=1}^{2^{m_2}(s-1) + s} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2}(s-1) + s} \left( \prod_{n=1}^{p+1} K(\ell_n; m_1 m_2; \alpha_1, \ldots, \alpha_n, \ell_{n+1}, \ell_{n}) \right) A_{i,j}^{(k+1)}
\]

\[
A_{i,j}^{(k+1)} = \sum_{\ell_1=1}^{2^{m_1}(s-1) + s} \sum_{\ell_2=1}^{2^{m_2}(s-1) + s} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2}(s-1) + s} \left( \prod_{n=1}^{p} (S_{i,j}^{(m_1,2^{m_2}+p,\alpha_1,\alpha_2,\ldots,\alpha_n,\ell_{n+1},\ell_{n})) \right) A_{i,j}^{(k+1)}
\]

\[
V_{i,j}^{(m_1,2^{m_2}+p,\alpha_1,\alpha_2,\ldots,\alpha_{p+1})} = \sum_{\ell_1=1}^{2^{m_1}(s-1) + s} \sum_{\ell_2=1}^{2^{m_2}(s-1) + s} \cdots \sum_{\ell_{p+1}=1}^{2^{m_2}(s-1) + s} \left( S_{i,j}^{(m_1,2^{m_2}+p,\alpha_1,\alpha_2,\ldots,\alpha_n,\ell_{n+1},\ell_{n})) \right) A_{i,j}^{(k+1)}
\]

The proof is complete.

### 4.2. Lower bound of entropy

In this subsection, the connecting operator \( C_{i,j}^{(m_1,2^{m_2}+p,\alpha_1,\alpha_2,\ldots,\alpha_{p+1})} \) is adopted to estimate the lower bound of entropy and in particular, to confirm that is positive. The following notation is used.

**Definition 4.7.** Let \( V = (V_1, \ldots, V_k)^T \), where \( V_k \) are \( N \times N \) matrices. Define the sum over \( V_k \) as

\[ |V| = \sum_{k=1}^{N} V_k. \]

If \( M = [M_{ij}] \) is a \( M \times M \) matrix, then

\[ |M| = \sum_{i=1}^{M} \sum_{j=1}^{M} M_{ij} V_j \]

Notably, (74) implies

\[ |V| = \sum_{k=1}^{M} A_{i,j}^{(k)} \]

\[ A_{i,j}^{(m_1,2^{m_2},\alpha_1,\alpha_2,\ldots,\alpha_{p+1})} \]

As is typical, the set of all matrices with the same order can be partially ordered.
Definition 4.8. Let $M = [M_{ij}]$ and $N = [N_{ij}]$ be two $M \times M$ matrices; $M \geq N$ if $M_{ij} \geq N_{ij}$ for all $1 \leq i,j \leq M$.

Notably, if $A_{2\times 2\times 2} \geq A'_{2\times 2\times 2}$, then $A_{2\times 2\times 2} \geq A'_{2\times 2\times 2}$ for all $m_2$, $m_3 \geq 2$. Therefore, $h(A_{2\times 2\times 2}) \geq h(A'_{2\times 2\times 2})$. Hence, the spatial entropy as a function of $A_{2\times 2\times 2}$ is monotonic with respect to the partial order $\geq$.

Definition 4.9. A $P + 1$ multiple index $A_P \equiv (a_{12} \cdots a_P, p_{a_{1}})$ (75)
is called a periodic cycle if

$$a_{p_{a_{1}} + 1} = a_{1},$$

(76)

where $1 \leq a_{1} \leq 2^{m_2}$ and $1 \leq p_{1} \leq P + 1$. It is called diagonal cycle if Eq. (76) holds and

$$a_{i} \in \{2^{m_2}(s-1) + 1 \mid s \leq 2^{m_1}\}$$

for each $1 \leq i \leq P + 1$. For a diagonal cycle Eq. (75)

$$a_{p_{a_{1}}} = a_{1}; a_{2}; \cdots; a_{P}$$

and

$$a_{p_{a_{1}}} = a_{P}; a_{P-1}; \cdots; a_{1}, \quad (n\text{-times})$$

First, prove the following lemma.

Lemma 4.10. Let $m_1 \geq 2$, $m_2 \geq 2$, $P \geq 1$, $A_P$ be a diagonal cycle. Then, for any $m_3 \geq 1$,

$$\rho(A_{m_1,m_2,m_3}^{(2\times 2\times 2)}) \geq \rho((S_{m_1,m_2,m_3}^{(2\times 2\times 2)} S_{m_1,m_2,m_3}^{(2\times 2\times 2)} \cdots S_{m_1,m_2,m_3}^{(2\times 2\times 2)})^{m_3} V_{m_1,m_2,2a_{1}}).$$

Proof. Since $A_P$ is a periodic cycle, Theorem 4.6 implies

$$V_{m_1,m_2,m_3}^{(2\times 2\times 2)} = (S_{m_1,m_2,m_3}^{(2\times 2\times 2)} S_{m_1,m_2,m_3}^{(2\times 2\times 2)} \cdots S_{m_1,m_2,m_3}^{(2\times 2\times 2)})^{m_3} V_{m_1,m_2,2a_{1}}.$$  (78)

Furthermore, $A_P$ is diagonal and $|V_{m_1,m_2,m_3}^{(2\times 2\times 2)} \alpha_{i_1} \alpha_{i_2} \alpha_{i_3}| = \alpha_{i_1} \alpha_{i_2} \alpha_{i_3}$ lies in the diagonal part of Eq. (73), with $m_3 + p = m_3 P + 2$. Accordingly,

$$\rho(h(A_{m_1,m_2,m_3}^{(2\times 2\times 2)} V_{m_1,m_2,2a_{1}})) \geq \rho(|V_{m_1,m_2,m_3}^{(2\times 2\times 2)} \alpha_{i_1} \alpha_{i_2} \alpha_{i_3}|).$$

(79)

Therefore, Eq. (77) follows from Eqs. (78) and (79).

The proof is complete.

The following lemma is useful in evaluating maximum eigenvalue of Eq. (77).

Lemma 4.11. For any $m_1 \geq 2$, $m_2 \geq 2$, $1 \leq k \leq 2^{m_2 - 1} m_2$ and $a_1 \in \{(s-1)2^{m_2} + 1 \leq s \leq 2 m_2\}$, if

$$\text{tr}(A_{m_1,m_2}^{(k)}) = 0,$$

then for all $1 \leq \ell \leq 2^{m_2 - 1} m_2$,

$$\rho(S_{m_1,m_2}^{(k)} S_{m_1,m_2}^{(k)} S_{m_1,m_2}^{(k)} \cdots S_{m_1,m_2}^{(k)}) = 0.$$  (80)

Proof. Since $A_{m_1,m_2}^{(k)}$ can be expressed as Eq. (68), $\text{tr}(A_{m_1,m_2}^{(k)}) = 0$ if and only if Eq. (80) holds for all $1 \leq \ell \leq 2^{m_2 - 1} m_2$. The second part of the Lemma 4.11 follows easily from the first part. The proof is complete.

By Lemmas 4.10 and 4.11, the lower bound of entropy can be determined as follows.

Theorem 4.12. Let $a_{12} \cdots a_P p_{a_1}$ be a diagonal cycle. Then, for any $m_1 \geq 2$, $m_2 \geq 2$,

$$h(A_{m_1,m_2}^{(2\times 2\times 2)}) \equiv \frac{1}{m_3 P} \log \rho(S_{m_1,m_2}^{(2\times 2\times 2)} S_{m_1,m_2}^{(2\times 2\times 2)} S_{m_1,m_2}^{(2\times 2\times 2)} \cdots S_{m_1,m_2}^{(2\times 2\times 2)} V_{m_1,m_2,2a_{1}}).$$

(81)

Proof. First, by the methods used to prove Lemmas 2.10 and 2.11 and Theorem 2.12 in [Ban et al., 2007],

$$\lim_{m_3 \to \infty} \frac{1}{m_3} \log \rho((S_{m_1,m_2}^{(2\times 2\times 2)} S_{m_1,m_2}^{(2\times 2\times 2)} \cdots S_{m_1,m_2}^{(2\times 2\times 2)})^{m_3} V_{m_1,m_2,2a_{1}})$$

$$= \log \rho(S_{m_1,m_2}^{(2\times 2\times 2)} S_{m_1,m_2}^{(2\times 2\times 2)} \cdots S_{m_1,m_2}^{(2\times 2\times 2)} V_{m_1,m_2,2a_{1}}).$$

(82)

is obtained. The detailed proofs are omitted here for brevity. Now,

$$h(A_{2\times 2\times 2}^{(2\times 2\times 2)}) \equiv \lim_{m_3 \to \infty} \frac{1}{m_3 P} \lim_{m_3 \to \infty} \frac{1}{m_3} \log \rho(|V_{m_1,m_2,2a_{1}}^{(2\times 2\times 2)} S_{m_1,m_2}^{(2\times 2\times 2)} S_{m_1,m_2}^{(2\times 2\times 2)} \cdots S_{m_1,m_2}^{(2\times 2\times 2)})^{m_3} V_{m_1,m_2,2a_{1}}).$$
is established. Indeed, from Eqs. (40) and (77),
\[
\begin{align*}
    h(A_2) &= \lim_{m_2 \to \infty} \frac{1}{(m_2P + 2)m_2} \log \rho(A_{2,m_2m_2}) \\
    &= \lim_{m_2 \to \infty} \frac{1}{m_2^{(m_2P + 2)m_2}} \log \rho(A_{1,m_2m_2}) \\
    &\geq \lim_{m_2 \to \infty} \frac{1}{m_2^{m_2P + 2}} \lim_{m_3 \to \infty} \frac{1}{m_3} \log \rho(S_{2,m_2m_2m_2,m_2,0}S_{2,m_2m_2m_2,0}S_{2,m_2m_2m_2,0}) \\
    &\cdots S_{2,m_2m_2m_2,0,0,0,0}1\right).
\end{align*}
\]

Apply Eq. (82) which completes the proof.  

**Remark 4.13.** By the similar method, the following lower bounds of entropy can also be estimated.

\[
\begin{align*}
    h(A_2) &\geq \lim_{m_2 \to \infty} \frac{1}{m_2^{m_2P + 2}} \log \rho(S_{2,m_2m_2m_2,m_2,0}S_{2,m_2m_2m_2,0}S_{2,m_2m_2m_2,0}) \\
    h(A_2) &\geq \lim_{m_2 \to \infty} \frac{1}{m_2^{m_2P + 2}} \log \rho(S_{2,m_2m_2m_2,m_2,0}S_{2,m_2m_2m_2,0}S_{2,m_2m_2m_2,0}) \\
    h(A_2) &\geq \lim_{m_2 \to \infty} \frac{1}{m_2^{m_2P + 2}} \log \rho(S_{2,m_2m_2m_2,m_2,0}S_{2,m_2m_2m_2,0}S_{2,m_2m_2m_2,0}) \\
    h(A_2) &\geq \lim_{m_2 \to \infty} \frac{1}{m_2^{m_2P + 2}} \log \rho(S_{2,m_2m_2m_2,m_2,0}S_{2,m_2m_2m_2,0}S_{2,m_2m_2m_2,0}) \\
    h(A_2) &\geq \lim_{m_2 \to \infty} \frac{1}{m_2^{m_2P + 2}} \log \rho(S_{2,m_2m_2m_2,m_2,0}S_{2,m_2m_2m_2,0}S_{2,m_2m_2m_2,0}) \\
\end{align*}
\]

The results in last three sections can be generated into p-symbols on Z^2×Z^2×Z^2 such as in two-dimensional case [Ban & Lin, 2005] and [Ban et al., 2007] and the details are omitted here for brevity.

5. Applications to 3DCNN

This section elucidates an interesting model in 3DCNN of the application of the method. The method is elucidated by considering 0,0,0 = α, α0,0,0 = α, α0,0,0 = α, which are nonzero; in other cases, α0,0,1 = α, and α1,0,0 = α, are zero. Then, the 3DCNN is of the form as in Eq. (6)

\[
\frac{du_{i,j,k}}{dt} = -u_{i,j,k} + w + af(u_{i,j,k}) + af(u_{i+1,j,k}) + af(u_{i,j+1,k}) + af(u_{i,j,k+1}).
\]

The stationary solution to Eq. (6) satisfies

\[
u_{i,j,k} = w + a_0w_{i+1}, j + a_0w_{i+1, j+1} + a_0w_{i+1, j+1, k} + a_0w_{i+1, j+1, k+1},
\]

for (i,j,k) ∈ Z^3 as in Eq. (7).

**Firstly,** consider the mosaic solution u = (u_{i,j,k}) to Eq. (7). If u_{i,j,k} ≥ 1, i.e. v_{i,j,k} = 1, then

\[
a_{1,j,k} (a - 1) + w + v_{i+1,j,k} + v_{i+1,j+1,k} + v_{i+1,j+1,k+1} + a_{i,j+1,k+1} \geq 0.
\]

If u_{i,j,k} ≤ 1, i.e. v_{i,j,k} = -1, then

\[
a_{1,j,k} (a - 1) - w - (a_{i+1,j,k} + v_{i+1,j,k} + a_{i+1,j,k+1}) \geq 0.
\]

Equation (7) has five parameters w, a, a0, a0, and a1. Three procedures are adopted to partition these parameters:

**Procedure (I).** The parameters a0, a0, a0, and a are initially expressed into three-dimensional coordinates, to solve Eqs. (83) and (84), as in Fig. 3.

Clearly 2^6 octants (I)–(VIII) exist in (a0, a0, a0) three-dimensional coordinates.

**Procedure (II).** In each octant are 3! relations

\[
\begin{align*}
    (i) & : |a_0| > |a_0| > |a_0| \\
    (ii) & : |a_0| > |a_0| > |a_0| \\
    (iii) & : |a_0| > |a_0| > |a_0| \\
    (iv) & : |a_0| > |a_0| > |a_0| \\
    (v) & : |a_0| > |a_0| > |a_0| \\
    (vi) & : |a_0| > |a_0| > |a_0| \\
\end{align*}
\]

(85)
Procedure (III). Each relation, denoted by $|a_1| > |a_2| > |a_3|$, two situations apply
\begin{enumerate}
  \item $|a_1| > |a_2| + |a_3|$  
  \item $|a_1| < |a_2| + |a_3|$.  
\end{enumerate}
(86)

However, in the $(a, w)$-planes, two sets of $2^3$ straight lines are important. The first set is
\[
\ell^+_a : (a - 1) + w + a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1} = 0,
\]
which is related to Eq. (83). The second set is
\[
\ell^-_a : (a - 1) - w - (a_x v_{i+1,j,k} + a_y v_{i,j+1,k} + a_z v_{i,j,k+1}) = 0,
\]
which is related to Eq. (84), where $v_{i+1,j,k}, v_{i,j+1,k}, v_{i,j,k+1} \in \{-1, 1\}$ and $1 \leq r \leq 8$. When $(a_x, a_y, a_z)$ lines in the open region (I)-(VIII), (I)-(VI) and (1)-(2) as in Fig. 3, Eqs. (85) and (86) are used to partition the $(w, a - 1)$-plane, as in Fig. 4.

The symbols $[m, n]$ in Fig. 4 have the following meanings. Consider, for example, $(a_x, a_y, a_z)$ lies in regions (VIII), (i) and (1) as in Fig. 3, Eqs. (85) and (86). This situation is expressed as (VIII)-(i)-(1), and considered $a_x < a_y < a_z < 0$.  

![Diagram](https://www.worldscientific.com/doi/abs/10.1142/S0218127408022021)
and $|a_x| > |a_y| + |a_z|$. Denoted by

<table>
<thead>
<tr>
<th>$c_i^1$</th>
<th>$c_i^2$</th>
<th>$c_i^3$</th>
<th>$c_i^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1^1$</td>
<td>$c_2^1$</td>
<td>$c_3^1$</td>
<td>$c_4^1$</td>
</tr>
<tr>
<td>$c_1^2$</td>
<td>$c_2^2$</td>
<td>$c_3^2$</td>
<td>$c_4^2$</td>
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<tr>
<td>$c_1^3$</td>
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<td>$c_3^4$</td>
<td>$c_4^4$</td>
</tr>
</tbody>
</table>

The only admissible patterns are exactly region (VIII)-(i)-(1) and  

$|a_xv_{i+1,j} + a_yv_{j+1,k} + a_zv_{i,j,k+1}|$

Then, $c_1^1 > c_2^1 > c_3^1 > c_4^1 > 0$. For instance, in  

region (VIII)-(i)-(1) and $|a_x|, |a_y|, |a_z| > 0$. The 24 local patterns  

can be produced. This fact is equivalent to the holding of inequalities in  

Eqs. (83) and (84) if and only if $v_{i,j,k}$, $v_{i+1,j,k}$, $v_{i,j+1,k}$, and $v_{i,j,k+1}$  

are of the form $c_1^1, c_2^1, c_3^1, c_4^1$. The corresponding transition  

matrices can be derived as  

$K_{x_2\times 2\times 2} = G \otimes E \otimes E \otimes E$.  

Then, according to Proposition 3.9, the admissible local patterns in  

$\Sigma_{x_2\times m_2\times m_3}$ and its corresponding  

transition matrices are  

$K_{x_2\times m_2\times m_3} = G \otimes (E \otimes E)^{m_2-1} \otimes (E \otimes E^2)$.
The spatial entropy can be exactly computed as
\[
\rho(S_{x,m_1,m_2,11}) = 2g^{m_2-1}
\]
where \( g = (1 + \sqrt{5})/2 \) is the golden-mean. Moreover, since
\[
A_{t;2,2} = G \otimes E \otimes E, \quad \text{then}
\]
\[
\rho(A_{t;2,2}) = \log g.
\]

Proof. According to Eq. (44),
\[
A_{t;2,2} = \mathcal{C}_{t;2,2}(x,m_2,1) \otimes (G^{m_2-1}) \otimes (E^{m_2})
\]
and
\[
\mathcal{R}(A_{t;2,2}) = 2m_2 + 1 \log (m_2 - 1),
\]
the spatial entropy can be exactly computed as
\[
\rho(A_{t;2,2}) = \log g.
\]

By Remark 4.3, the connecting operator
\[
\mathcal{C}_{t;2,2}(x,m_2,1) + \mathcal{R}(A_{t;2,2}) = (G^{m_2-1}) \otimes E.
\]

Therefore, based on Remark 4.13, the lower bound of spatial entropy is estimated as
\[
h(A_{t;2,2}) \geq \lim_{m_2 \to \infty} \frac{1}{2m_2} \log \rho(S_{x,m_1,m_2,11})\]
\[
= \lim_{m_2 \to \infty} \frac{2g^{m_2-1}}{2m_2}
\]
\[
= \frac{1}{2} \log g.
\]

Remark 5.2. For the general template \( A = (a_{t;2,2}) \) where \( a_{t;2,2} \neq 0 \), the basic set in \( \Sigma_{t;2,2} \) must be extended to the basic set in \( \Sigma_{t;4,4} \). Then, the method described above can be applied, as stated in Remark 4.14. The details are omitted here for brevity.

References


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