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Willmore P-泛函
Willmore P-Functional

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摘要

假設 M 為 3-維單位球面上之一 Willmore 曲面. 此報告首先建立相關積分恆等式與不等式, 再利用這些積分恆等式與不等式，經由 Willmore 泛函的已知結果, 得到 Willmore 2-泛函之下界估計。另一方面，利用這些積分恆等式得以改良前人一特定化 M 之點態估計。

關鍵詞 Willmore 曲面、球面

Abstract

Let M be a Willmore surface in the 3 dimensional unit sphere. Using the Willmore equation and a formula related to the trace free tensor of the second fundamental form, we find certain integral identities and inequalities. It follows from these identities and inequalities and well known lower bound estimates for the Willmore functional, we obtain certain estimates for the Willmore 2-functional. On the other hand, we improve a pointwise estimate for classification of Willmore surfaces.

Keywords: Willmore surfaces, spheres.

1. Introduction

Let M be a compact surface in the 3-dimensional unit sphere S^3. Denote by [h_{ij}] the second fundamental form of M, and the mean curvature of M by $H = \sum h_{ii}$. Let $\phi_{ij} = h_{ij} - \frac{H}{2}\delta_{ij}$ and $\Phi = \sum \phi_{ij}^2$ the square length of the trace free tensor. Then the Willmore p-functional is given by

$$W_p(M) = \int_M \Phi^p.$$

When p = 1, this functional $W(M) = W_1(M)$ is invariant under conformal transformations of S^3 , which is the Willmore functional. The Willmoe conjecture says that $W(M) \geqq 4\pi^2$ holds for for all immersed tori $x : M \to S^3$. A surface in S^3 is called a Willmore surface if it is a critical surface of the Willmore functional; A surface M in S^3 is a Willmore surface if and only if

$$\Delta H + H\Phi = 0.$$

It is obvious that all minimal surface in S^3 are Willmore surface. Pinkall constructed many nonminimal Willmore surface in S^3 (see [P]). Weiner conjecture says that the only closed orientable immersed surfaces in S^3 whose centroids is 0 is a minimal surfaces in S^3 (see [W]).

In the case of minimal surfaces, it is well known that if $0 \leqq \Phi \leqq 2$, then M is either the equatorial sphere or a Clifford torus (see [CCK]). This result was extend to class of Willmore surfaces by Li (see [L1]): If M is a Willmore surface in S^3 with $0 \leqq \Phi \leqq 2$, then M is either totally umbilic or a Clifford torus. On the other hand, Topping proved that if M is a immersed torus in S^3 with K 0 (i.e., 4+ H^2=2Φ), then W(M)$\geqq 4\pi^2$ (see [T]).

Our first result is a improvement of a result of Li (see [L1])

If M is a Willmore surface in S^3 with $0 \leqq \Phi \leqq 2+H^2/4$, then M is either totally umbilic or a Clifford torus.

In fact, we have an analogue result for higher codimension case.

In the case of minimal surfaces, we proved that if the L^2 norm of Φ satisfies the global pinching condition $\|\Phi\|_2 \le 2\sqrt{2}®$, then M is either the equatorial sphere or a Clifford torus ([H]). Here we consider the Willmore p functional in the case of p = 2. The main step of this report is to find the following identities for Willmore surfaces

$$0 = \int_M \Phi(2 + \frac{H^2}{2} - \Phi) - \frac{|\nabla H|^2}{2} + ¤^2_{ijk},$$

$$\int_M \Phi^2 = \int_M 2\Phi + ¤^2_{ijk}$$

$$= 16®(g-1) + \int_M 2(2 + \frac{H^2}{2}) + ¤^2_{ijk},$$

where g is the genus of M. It follows that corresponding to each lower bound estimate of the Willmore functional, there is a lower bound estimate of the Willmore 2-functional. There were many lower bound estimates of the Willmore functional (Li and Yau, Montiel and Ros, …etc). As applications, we have

If M is a Willmore surface in the 3 dimensional unit sphere , then $\|\Phi\|_2 \ge 4\sqrt{®g}$, where g is the genus of M. Moreover if $\|\Phi\|_2 \le 4\sqrt{®g}$, then M is totally umbilic.

and

If M is a Willmore torus in the 3 dimensional unit sphere, then $\|\Phi\|_2 \ge \sqrt{a_1 A}$, where a_1 is the first eigenvalue of the Laplacian and A is the area of M.

In general, one believes that the Willmore equation does not appear to encode enough information since the class of all Willmore surfaces turns out to be rather large and difficult to control. However, we see that the Willmore equation plays an important role when we consider Willmore p-function.

2. The Willmore Flow

In this section we state some basic equations by the notation of moving frame. Let M be a compact surface in the 3-dimensional unit sphere S^3. Denote by e_1, e_2, x and N, where e_1, e_2 are tangent to M, x is the position vector of M and N is the unit normal of M in S^3. Let $ù_1$ and $ù_2$ be the dual coframe. Then the structure equations are

$$\begin{aligned}
dx &= \Sigma\mu_i e_i, \\
de_i &= \Sigma\mu_{ij} e_j + h_{ij}\mu_j N - \mu_i x, \\
dN &= -\Sigma h_{ij}\mu_j e_i, \\
d\mu_i &= \Sigma\mu_{ij} \wedge \mu_j, \\
d\mu_{ij} &= \Sigma\mu_{ik} \wedge \mu_{kj} - \frac{1}{2} R_{ijkl}\mu_k \wedge \mu_l, \\
R_{ijkl} &= ¢_{ik}¢_{jl} - ¢_{il}¢_{jk} + h_{ik}h_{jl} - h_{il}h_{jk}.
\end{aligned}$$

Suppose that the evolution equation for the Willmoe functional is given by x_t = f N. Then we have

$$x_t = f\,N,$$
$$(e_i)_t = \Sigma\, a_{ij}\, e_j + f_i\, N,$$
$$N_t = -f\,x - \Sigma\, f_i\, e_i,$$
$$(\mu_i)_t = \Sigma\, a_{ij}\,\mu_j - fh_{ij}\,\mu_{\mathrm{j}},$$

where a_{ij} are skew symmetry. It follows that

$$(\mu_1 \wedge \mu_2)_t = -fH\mu_1 \wedge \mu_2,$$
$$(h_{ij})_t = f_{ij} + f¢_{ij} - h_{ik}a_{kij} - h_{kij}a_{ki} + fh_{ik}h_{kj.}$$

Thus if x(.,t) is the gradient flow of the willmore functional,then

$$f = -(2\Delta H + 2H\Phi).$$

We need the following Lemmas. Lemmas 1 and 2 are straightforward computation.

Lemma1.

$$\frac{1}{2}\Delta\Phi = \Sigma ¤_{ij} H_{ij} + \Phi(2 + \frac{H^2}{2} - \Phi) + \Sigma ¤_{ijk}{}^2.$$

Lemma 2.

$$¤_{ijk} = ¤_{ikj} + \frac{H_j}{2}¢_{ki} - \frac{H_k}{2}¢_{ij} \text{ for all i, j, k.}$$

The following Lemma is the Gauss-Bonnet theorem.

Lemma 3.

$$\int_M (2 + \frac{H^2}{2} - \Phi) = 8®(g-1).$$

We can check that

Lemma 4.

$$\Phi\Sigma ¤_{ijk}{}^2 = \frac{1}{2}|\nabla\Phi|^2 + \frac{1}{2}\Phi|\nabla H|^2 - \Sigma ¤_{ij} H_i \Phi_j.$$

It follows from the Willmore equation that

Lemma 5.

$$\int_M |\nabla H|^2 = \int_M H^2\Phi.$$

Using Lemma 2, we have the following inequality

Lemma 6.

$$\Sigma ¤_{ijk}{}^2 \geq \frac{1}{4}|\nabla H|^2.$$

The following relations are crucial which follows from the above relations.

Lemma 7.

$$0 = \int_M \Phi(2 + \frac{H^2}{2} - \Phi) - \frac{|\nabla H|^2}{2} + \Sigma ¤_{ijk}^2.$$

Lemma 8.

$$\int_M \Phi^2 = \int_M 2\Phi + \Sigma ¤_{ijk}^2$$
$$= 16®(g-1) + \int_M 2(2 + \frac{H^2}{2}) + \Sigma ¤_{ijk}^2.$$

Lemma 9. If $\Phi>0$ on M, then g = 1.

Lemma 9 follows from Lemma 4.

3. Proofs.

We prove that if M is a Willmore surface in S^3 with $0 \leqq \Phi \leqq 2+H^2/4$, then M is either totally umbilic or a Clifford torus.

Proof. It follows from Lemma 1 that

$$0 = \int_M \Sigma ¤_{ij} H_{ij} + \Phi(2 + \frac{H^2}{2} - \Phi) + \Sigma ¤_{ijk}{}^2$$
$$= \int_M -\Sigma ¤_{ijj} H_i + \Phi(2 + \frac{H^2}{2} - \Phi) + \Sigma ¤_{ijk}{}^2.$$

Since

$$\Sigma ¤_{ijj} = \Sigma ¤_{ijj} = \Sigma ¤_{jji} + \frac{H_i}{2} = \frac{H_i}{2},$$

Lemma 6 implies

$$0=\int_M -\frac{1}{4}|\nabla H|^2+\Phi(2+\frac{H^2}{2}-\Phi)$$
$$=\int_M \Phi(2+\frac{H^2}{4}-\Phi).$$

If $0\leqq\Phi\leqq 2+H^2/4$, then either $\Phi=0$ or $\Phi=2+H^2/4$.

If $\Phi=0$, then M is totally umbilic. If $\Phi=2+H^2/4$, then $\Phi>0$ on M, Lemma 9 implies that M is a torus. Furthermore, Since

$$0=\int_M (2+\frac{H^2}{2}-\Phi)$$
$$=\int_M \frac{H^2}{4},$$

M is a minimal surface. Being a minimal surface with the square of length of the second fundamental form S = 2, M is a Cifford torus. □

Since

$$\int_M \Phi^2=\int_M 2\Phi+\Sigma\varpi_{ijk}^2$$
$$\geq\int_M 2\Phi,$$

the lower bound estimates for the L^2 norm of Φ follows immediately from certain well known estimates of Willmore functional.

As a final remark, we must mention that a lower bound estimate without using any estimates of Willmore functional is more interesting.

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