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A pinching theorem for conformal classes of Willmore surfaces in the n-sphere

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摘要

假設 M 為 n 維球面上之 Willmore 緊緻曲面。此報告旨在證實下列保角估計成立:
若當 n=3 時，

$$\inf_{g\in G}\max_{g\circ x(M)}(\Phi_g-\frac{1}{4}H_g^2)\le 2,$$

當 n≧4 時，

$$\inf_{g\in G}\max_{g\circ x(M)}(\Phi_g-\frac{1}{8}H_g^2-\sqrt{\frac{4}{9}+\frac{1}{6}H_g^2+\frac{1}{96}H_g^4})\le\frac{2}{3},$$

其中 G 為 n 維球面之保角群，Φ為第二基本形之零跡張量長之平方，H 為均曲率向量之長，則 M 為 totally umbilic 球或保角的 Clifford 環面或保角的 Veronese 曲面。

關鍵詞 Willmore 曲面、球面、第二基本形

Abstract

Let $x: M\rightarrow S^n$ be a compact immersed Willmore surface in the n-dimensional unit sphere. In this report, we prove that if

$$\inf_{g\in G}\max_{g\circ x(M)}(\Phi_g-\frac{1}{4}H_g^2)\le 2, \text{ when } n=3,$$

$$\inf_{g\in G}\max_{g\circ x(M)}(\Phi_g-\frac{1}{8}H_g^2-\sqrt{\frac{4}{9}+\frac{1}{6}H_g^2+\frac{1}{96}H_g^4})\le\frac{2}{3}, \text{ when } n\ge 4,$$

where G is the conformal group of the ambient space, Φ_g and H_g are the square of the length of the trace free part of the second fundamental form and the length of the mean curvature vector of the immersion $g\circ x$ respectively, then x(M) is either a totally umbilic sphere or a conformal Clifford torus or a conformal Veronese surface.

Keywords: Willmore surfaces, spheres, the second fundamental form

1. Introduction

Let $x: M\rightarrow S^n$ be a compact immersed surface in the n-dimensional unit sphere S^n. Denote by h_{ij}^{α} the components of the second fundamental form of M, by $H^{\alpha}=\sum h_{ii}^{\alpha}$ the α-component of the mean curvature vector and by H the length of the mean curvature vector,

$H=\sqrt{\sum(H^{\alpha})^2}$. Let $\phi_{ij}^{\alpha}=h_{ij}^{\alpha}-\frac{H^{\alpha}}{2}\delta_{ij}$ and $\Phi=\sum(\phi_{ij}^{\alpha})^2$ the square length of the trace free tensor. Then the Willmore functional of X is given by

$$W(X)=\int_M \Phi,$$

where the integration is with respect to the area measure of M. This functional is preserved if we move M via conformal transformations of S^n. The critical points of the Willmore functional W are called Willmore surfaces. They satisfy the Euler-Lagrange equation

$$\Delta H^{\alpha} + \Sigma \phi_{ij}^{\alpha} \phi_{ij}^{\beta} H^{\beta} = 0,$$

where Δ is the Laplacian in the normal bundle

NM (see [Ch]). Thus any minimal surface in S^n is a Willmore surface. The set of Willmore surfaces turns out to be larger than that of minimal surfaces.

For M being a minimal submanifold in the n-dimensional unit sphere, there are vast estimates for the square of the length of the second fundamental form. Significant works in this direction has been obtained by Simons (see [S]), Chern, do Carmo and Kobayashi (see [CDK]), Peng and Terng (see [PT]) etc. One expects that similar estimates are also valid for generalized Willmore submanifolds. Based on this observation, Li proved that if M is a compact Willmore surface in the n-dimensional unit sphere, the square of the length of the trace free part of the second fundamental form satisfying certain inequality, then M is the totally umbilical sphere, or the Clifford torus, or the Veronese surface (see [L1], [L2] and [L3]). It is remarkable that the Clifford torus and Veronese surface are minimal surfaces except totally umbilical spheres.

For M being a hypersurface with constant mean curvature in the n-dimensional unit sphere S^n. Alencar and do Carmo obtained a pinching constant which depends on the mean curvature (see [AD]). For higher codimension, Santo was able to generalize this result to submanifolds with parallel mean curvature vector (see [Sa]). It is interesting to find an upper estimate for Φ including the mean curvature because in general a Willmore surface is not minimal. Our starting point is to find such an estimate. The second estimate improves an estimate given previously by the author (see [CH]).

Theorem A

Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n.

If n = 3, $\Phi \le 2 + \frac{1}{4}H^2$ on M, *then either Φ = 0 and M is totally umbilical or* $\Phi = 2 + \frac{1}{4}H^2$. *In the latter case, M is the Clifford torus.*

If $n \ge 4, \Phi \le \frac{2}{3} + \frac{1}{8}H^2 + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4}$, *then either Φ = 0 and M is totally umbilical or* $\Phi = \frac{2}{3} + \frac{1}{8}H^2 + \sqrt{\frac{4}{9} + \frac{1}{6}H^2 + \frac{1}{96}H^4}$. *In the latter case, M is the Veronese surface.*

Because of the fact that the Willmore functional is conformal invariant, it is more interesting to find an estimate concerning the conformal classes of Willmore surfaces. It is remarkable that the Clifford torus and the Veronese surface are the minimal surface in the 3 and 4 dimensional unit spheres (see [CDK]). Just as the results of Li, Theorem A does not characterize any non-minimal Willmore surface except the totally umbilic spheres. However, these estimate is sharp in the sense that for every given positive ε, there is a compact Willmore surface M in S^3 and

S^4 satisfying $0< \Phi\le 2+\frac{1}{4}H^2+\varepsilon$ and $0<\Phi\le\frac{2}{3}+\frac{1}{8}H^2+\sqrt{\frac{4}{9}+\frac{1}{6}H^2+\frac{1}{96}H^4}+\varepsilon$ respectively which are not the Clifford torus and the Veronese surface.

For characterizing a non-minimal Willmore surface, for each immersion x of M into the unit n-sphere S^n, we consider the infimum of maximum values of $\Phi-\frac{1}{4}H^2$ if n = 3 and $\Phi-\frac{1}{8}H^2-\sqrt{\frac{4}{9}+\frac{1}{6}H^2+\frac{1}{96}H^4}$ if $n\ge 4$ obtained by composition of x with g where g ranges over all conformal mappings of S^n. This conformal invariant depends on the immersion x. We show that this conformal invariant characterizes the totally umbilic sphere and the conformal classes of the Clifford torus and the Veronese surface. Since the conformal group G of the ambient space S^n is not compact, the proof involves some new tricks. The following is the main result of the report. .

Theorem B

Let M be a compact immersed Willmore surface in the n-dimensional unit sphere S^n. If n=3 and $\inf_{g\in G}\max_{g\circ x(M)}(\Phi_g-\frac{1}{4}H_g^2)\le 2,$

then either Φ = 0 and M is totally umbilical or M is a conformal Clifford torus.

If $n\ge 4$ and $\inf_{g\in G}\max_{g\circ x(M)}(\Phi_g-\frac{1}{8}H_g^2-\sqrt{\frac{4}{9}+\frac{1}{6}H_g^2+\frac{1}{96}H_g^4})\le\frac{2}{3},$

then either Φ = 0 and M is totally umbilic or M is a conformal Veronese surface, where G is the conformal group of the ambient space S^n, Φ_g and H_g *are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g\circ x$ respectively.*

Theorem A characterizes the totally umbilic spheres, the Clifford torus and the Veronese surface by use of an integral inequality in terms of Φ and H. The conformal estimate is dealt in Theorem B. The main idea in the proof of Theorem B follows very closely the proof we gave of Theorem A. In this proof, we consider a minimizing sequence $\{g_m\}$ in G. If this minimizing sequence is convergent in G, the assertion follows from Theorem A. Otherwise, we will show that M must be totally umbilic. The proof requires some careful modifications in progress.

In this report, we will give the outline of the proof of Theorem B in the case of codimension one. The proof of higher codimension is more complicated.

2. The proof of Theorem B

In this section we show briefly the proof of Theorem B in the case of codimension one.

Step 1. By the hypothesis, there is a sequence g_m in G such that $\Phi_m-\frac{1}{4}H_m^2\le 2+\frac{1}{m}$ on M, for all m=1,2,… , where Φ_m and H_m are the square of the length of the trace free part of the second fundamental form and the mean curvature of the immersion $g_m\circ x$, respectively. Without loss of generality, we may assume that g_m in the open unit disk D_4. The closure of D_4 being compact, there exists a convergent subsequence . We may assume that g_m converges to g_0 for some g_0 in the closed unit disk. If g_0 in D_4 , then the desired conclusion follows from Theorem A. So we need only consider the case that g_0 is a constant unit vector. In this case we will show below that M is totally umbilic.

Step 2. Suppose, to get a contradiction, that Φ is positive somewhere on M. To avoid ambiguity, we shall now use the notations da and da_m for the area measures of x and $g_m \circ x$, respectively. Since $g_m \circ x$ are Willmore surfaces, the integral inequality of Theorem A gives

$$2\int_M \Phi_m da_m \le \int_M \Phi_m(\Phi_m - \frac{H_m^2}{4})da_m$$
$$\le (2+\frac{1}{m})\int_M \Phi_m da_m.$$

Since Willmore functional is invariant under conformal transformations of S^3 ,

$$2(1-|g_m|^2)\int_M \Phi da$$
$$\le \int_M \Phi[(1+\langle x,g_m\rangle)^2\Phi - \frac{1}{4}((1+\langle x,g_m\rangle)H + 2\langle e_3,g_m\rangle)^2]da$$
$$\le (2+\frac{1}{m})(1-|g_m|^2)\int_M \Phi.$$

Letting m $\to$ ∞, we find that

$$\int_M \Phi[(1+\langle x,g_0\rangle)^2\Phi - \frac{1}{4}((1+\langle x,g_0\rangle)H + 2\langle e_3,g_0\rangle)^2]da = 0.$$

On the other hand, since $\Phi_m - \frac{H_m^2}{4} \le 2 + \frac{1}{m}$ on M, we have

$$(1+\langle x,g_m\rangle)^2\Phi - \frac{1}{4}((1+\langle x,g_m\rangle)H + 2\langle e_3,g_m\rangle)^2$$
$$\le (2+\frac{1}{m})(1-|g_m|^2).$$

When m tends to infinity, we find that $(1+\langle x,g_0\rangle)^2\Phi - 1/4((1+\langle x,g_0\rangle)H+2\langle e_3,g_0\rangle)^2$ is nonpositive on M. We then conclude that $(1+\langle x,g_0\rangle)^2\Phi = 1/4((1+\langle x,g_0\rangle)H+2\langle e_3,g_0\rangle)^2$ or $\Phi = 0$ on M, and hence $(1+\langle x,g_0\rangle)^2\Phi = 1/4(1+\langle x,g_0\rangle)F^2$ provided $\Phi > 0$, where $F=(1+\langle x,g_0\rangle)H+2\langle e_3,g_0\rangle$. This implies that either $F=2(1+\langle x,g_0\rangle)\sqrt{\Phi}$ or $F= - 2(1+\langle x,g_0\rangle)\sqrt{\Phi}$ on each of the connected components of the set of points where $\Phi > 0$.

Step3. For each fixed m, let $\overline{x} = g_m \circ x$. Since $g_m \circ x$ is a Willmore immersion, as the proof of Theorem A, we have the following equation again

$$0 = \int_M \sum \overline{\phi_{ijk}^2} + \overline{\phi_{ij}\ H_{ij}} + \overline{\Phi}(2+\frac{\overline{H^2}}{2} - \overline{\Phi})\,d\overline{a}$$
$$= \int_M \sum \overline{\phi_{ijk}^2} - \overline{\phi_{ijj}\ H_i} + \overline{\Phi}(2+\frac{\overline{H^2}}{2} - \overline{\Phi})\,d\overline{a}$$
$$= \int_M \sum \overline{\phi_{ijk}^2} - \frac{|\overline{\nabla H}|^2}{2} + \overline{\Phi}(2+\frac{\overline{H^2}}{2} - \overline{\Phi})\,d\overline{a}.$$

When m tends to infinity, it follows that

$$0 = \int_M \sum \psi_{ijk}^2 - \frac{1}{2}|\nabla F|^2. + \Phi(\frac{1}{2}F^2 - (1+\langle x,g_0\rangle)^2\Phi)$$
$$= \int_M \sum \psi_{ijk}^2 - \frac{1}{2}|\nabla F|^2. + \frac{1}{4}\Phi F^2$$
$$\ge \int_M (\psi_{111}+\psi_{122})^2 + (\psi_{211}+\psi_{222})^2 - \frac{1}{2}|\nabla F|^2 + \frac{1}{4}\Phi F^2$$
$$\ge \int_M -\frac{1}{4}|\nabla F|^2 + \frac{1}{4}\Phi F^2$$
$$= 0,$$

here we use the identity $(1+\langle x,g_0\rangle)^2\Phi^2 = 1/4\Phi F^2$. Therefore we have $\psi_{111}=\psi_{122}$ and $\psi_{211}=\psi_{222}$. Combining the last two equations with certain properties of ψ_{ijk} and simplifying, we can express ψ_{ijk} in terms of F_1 and F_2

$$\psi_{111} = \psi_{122} = \psi_{212} = -\psi_{221} = \frac{1}{4}F_1$$

and

$$\psi_{121} = \psi_{211} = \psi_{222} = -\psi_{112} = \frac{1}{4}F_2.$$

Step 4. Let $U = 2(1+ \langle x,g_0 \rangle)\sqrt{\Phi}$, and let Ω be a connected component of the set of points where Φ is positive. Then

$$U_1 = 2\sqrt{\Phi}\langle e_1,g_0\rangle + 4\frac{\phi_{11}}{\sqrt{\Phi}}(1+\langle x,g_0\rangle)\phi_{111} + 4\frac{\phi_{12}}{\sqrt{\Phi}}(1+\langle x,g_0\rangle)\phi_{121}$$
$$U_2 = 2\sqrt{\Phi}\langle e_2,g_0\rangle + 4\frac{\phi_{11}}{\sqrt{\Phi}}(1+\langle x,g_0\rangle)\phi_{112} + 4\frac{\phi_{12}}{\sqrt{\Phi}}(1+\langle x,g_0\rangle)\phi_{122}$$

on Ω. Since ψ_{ijk} can be expressed in terms of F_1 and F_2, we then obtain that for all i,

$$U_i = \sum \frac{\phi_{ij}}{\sqrt{\Phi}}F_j$$

on Ω. Therefore we have $[\nabla U]^2 = \frac{1}{2}|\nabla F|^2$ on Ω. On the other hand, we know that $U = \pm F$ on Ω, $[\nabla U]^2 = |\nabla F|^2$ on Ω. We then conclude that the gradient of F vanishes on Ω, and hence F is a constant on Ω. Since every immersion is locally an embedding, 1+<x,g_0> vanishes only at most finite points on M, and $(1+<x, g_0>)^2\Phi^2=\frac{1}{4}\Phi F^2$ on M, this constant must be nonzero by the continuity of Φ. Since F is a nonzero constant satisfying the equation $\Delta F+\Phi F=0$, Φ vanishes on Ω, we get a contradiction. This contradiction shows that Φ vanishes identically, and M is totally umbilical. This completes the proof of Theorem B in the case of codimension one.

The main idea for the case higher codimension follows very closely the proof given above if we consider the weaker pinching condition $\inf_{g\in G}\max_{g\circ x(M)}(\Phi_g-\frac{1}{6}H_g^2)\leq\frac{4}{3}$. However, the proof of Theorem B for higher codimension is more complicated.

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