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Bent 函數及其相關的強正則圖

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布林函數 Walsh 質譜的計算的複雜度，一般而言十分困難。然而某些特定族類的該函數的值譜所具有的特殊性質，能有效降低其複雜度。值得一提的是布林函數 \( f \) 的值譜分析可轉化為凱氏圖 Cayley Graphs 的質譜相關問題，因而代數圖論在 Bent 函數的研究裏，伴演一個十分積極的角色，由其是相異係數不大時為然。我們據以得到其所對應的強正則圖的參數及其值譜；並且列出點數不超過 280 的所有可能情形。

The complexity of computing the Walsh spectrum of Boolean functions is difficult in general, however several interesting classes of such functions have a very special spectrum, whose ad hoc computation can be carried out significantly faster than in the general case. It is worth to note that the spectral analysis of Boolean functions can be viewed as a Cayley graph eigenvalue problem, this observations allow the using of tools from algebraic graph theory for investigations related to the spectral coefficients of Boolean functions, especially when the number of distinct coefficients is small. The main motivation for introducing the graph \( G_f \) is that its spectrum coincides with the Walsh spectrum of its associated Boolean function \( f(x) \). This brings the problem of analyzing the spectral coefficients of Boolean functions into the framework of spectral analysis of graphs, i.e., it makes it possible to use techniques from graph spectra for the evaluation of spectral coefficients. More precisely, the results from algebraic graph theory can be applied to analyze Boolean functions with a few distinct spectral coefficients, the fewer is the number of distinct coefficients, the stronger are the algebraic properties of the function; this leads to a nice interpretation for the well-known class of bent functions in terms of strongly regular graphs.
Strongly Regular Graphs associated with Bent Functions

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1. Bent Functions

The Fourier transform of a Boolean function $g(x) : Z_2^n \rightarrow Z_2$ is defined to be

$$g^*(\lambda) = \frac{1}{2^n} \sum_{x \in Z_2^n} g(x) \cdot (-1)^{\langle \lambda, x \rangle}.$$ 

It is known that $g(x) = \sum_{\lambda \in Z_2^n} g^*(\lambda) \cdot (-1)^{\langle \lambda, x \rangle}$. A Boolean function $f : Z_2^n \rightarrow Z_2$ is called a bent function if $\left((-1)^{f(x)}\right)(\lambda) = \pm \frac{1}{\sqrt{2^n}}$ for any $\lambda \in Z_2^n$, the term of bent was coined by Rothaus [9].

Theorem [9]

If $f(x)$ is a bent function on $Z_2^n$ with $n \geq 3$, then $n = 2k$ must be even, and the degree of $f(x)$ is at most $k$; moreover $f(x)$ is irreducible whenever $\deg(f(x)) = k \geq 3$.

Some basic properties of bent functions together with their relationships with some combinatorial structures are summarized in the following theorem. The Boolean function $f(x)$ is bent if and only if the matrix $\left[(-1)^{f(x+y)}\right]$ is a Hadamard matrix. The Fourier
transform of a bent function is again a bent function.

2. The Cayley Graphs associated with Bent Functions

The Cayley graph $G_f = (V(f), E_f)$ associated with a Boolean function $f : Z_2^n \rightarrow Z_2$ is defined on the vertex set $V(f) = Z_2^n$, with $u, w \in Z_2^n$ adjacent if $w \oplus u \in \Omega_f = f^{-1}(1)$, or equivalently $f(w \oplus u) = 1$. The graph $G_f$ is $|\Omega_f|$-regular with $2^{n-\dim(\Omega_f)}$ connected components, the graph $G_f$ is connected if $\dim(\Omega_f) = n$. The spectrum of $G_f$ is usually denoted by $\text{Spec}(G_f) = \left(\frac{1}{|\Omega_f|}, \lambda_1, ..., \lambda_{2^n-1}\right)$ where $\lambda_i = \sum_{x \in Z_2^n} f(x)(-1)^{(b(i), x)} = 2^n \cdot f^*(b(i))$

Upper and lower bounds on the rank (over the real field) of the adjacency matrix $A_f$ of $G_f$ i.e., the number of nonzero spectral coefficients of the function $f$, are given in terms of degrees of polynomials representation of $f$. Some properties of the Fourier coefficients and its associated Cayley graphs are given in the following.

Theorem [1] If $f : Z_2^n \rightarrow Z_2$, and $\lambda_i, 0 \leq i \leq 2^n - 1$, are the eigenvalues of the graph $G_f$, then

a. $\lambda_i = 2^n f^*(b(i))$ for $0 \leq i \leq 2^n - 1$;

b. the multiplicity of its largest eigenvalue $f^*(b(0))$ is $2^{n-\dim(\Omega_f)}$ (which implies the graph $G_f$ is $|\Omega_f|$-regular with $2^{n-\dim(\Omega_f)}$ connected components and the graph $G_f$ is connected if $\dim(\Omega_f) = n$);

A Boolean function is characterized by its spectrum if it is possible to identify its associated graph (i.e., determine all the details of its topology) only on the basis of the knowledge of its distinct eigenvalues, i.e., without using any information regarding their eigenvectors. It is interesting to note that the fewer the number of distinct spectral coefficients are, the stronger are the algebraic properties of the set $\Omega_f$; for instance, it is well-known that if a connected graph has exactly $m$ distinct eigenvalues, then its diameter $d$ satisfies $d \leq m - 1$.

A $k$-regular graph $G$ is strongly regular if there exist nonnegative integers $a$ and $c$ such that for all vertices $u, v$, the number $|G_i(u) \cap G_i(v)|$ of vertices adjacent to both $u$ and $v$ is $a$ if $u$ and $v$ are adjacent, and $c$ otherwise. A $k$-regular connected graph is strongly regular if and only if it has exactly three distinct eigenvalues $\lambda_0 = k, \lambda_1, \lambda_2$. A rephrase of Parseval’s identity gives that $f^*(b(0)) = \sum_{i=0}^{2^n-1} (f^*(b(i)))^2$ and then yields the following useful quality $(k - \lambda_1)(k - \lambda_2) = 2^r(k + \lambda_1 \lambda_2)$ where $k = |\Omega_f|$, and $r$ must be replaced by $\dim(\Omega_f)$ if $G$ is not connected. If $G$ is strongly regular, then $a = k + rs + r + s$ and $c = k + rs$. It was also observed that the class of bent functions is associated to a very special class of strongly regular graphs indeed exactly identifies the bent functions.
Theorem [1]
If $G_f$ is a connected strongly regular graph, then there exists $v \in \Omega_f$ such that $u \oplus v \in \Omega_f$ for each $u \in Z_2^n \setminus \Omega_f$, and there exist $h$ elements $w \in \Omega_f$ such that $v \oplus w \in \Omega_f$, where $h = e$ if $v \in \Omega_f$, and $d$ if $v \not\in \Omega_f$. For each $v \in Z_2^n$,

In order to find a complete characterization of the class of functions with three distinct nonzero spectral coefficients and with the additional property $a = c$, we are then left with the problem of understanding whether or not there exists other integer solutions to $x^2 - 2^n x + (2^n - 1)y^2 = 0$. It was proved in [2] that the equation has integer solutions in $x$ and $y$ only if $y^2 = 0$, $1$, $2^{n-2}$. As a consequence, bent functions can be characterized as binary functions with a certain class of strongly regular graphs.

Theorem [1,2]
a. The associated Cayley graph $G_f$ of a bent function is a strongly regular graph $SRG(v, k, \lambda, \lambda)$.

b. The bent functions are the only binary functions $f$ whose associated graph $G_f$ is a strongly regular graph $SRG(v, k, \lambda, \lambda)$.

Those graphs $G_f$ with small numbers of distinct eigenvalues are considered: if $G_f$ has a single eigenvalue, then $G_f = K_{2^n-1}$; if $G_f$ has two distinct eigenvalues, then either $G_f = 2^n - |K_{b^1}| \Sigma$ or $G_f = 2^n - |K_{b^1}| \Sigma$ with loops otherwise; if $G_f$ has three eigenvalues, then

a. $(\lambda_0, \lambda_1, \lambda_2) = (|\Omega_f|, 0, |\Omega_f|)$ if and only if $G_f$ is the complete bipartite graph between vertices in $\Omega_f$ and in $Z_2^n \setminus \Omega_f$.

b. $(\lambda_0, \lambda_1, \lambda_2) = (|\Omega_f|, 0, \lambda_2)$ if and only if $G_f$ is a complete multipartite graph with $G_f = (-\frac{|\Omega_f|}{\lambda_2} + 1)K_{\lambda_2}$ and with $Spec(G_f) = ((2^{n-1})^0, (0)(2^{n-1})^0, (\lambda_2)(\frac{-2^{n-1}}{\lambda_2})^0)$

c. if $G_f$ is connected, then $G_f$ is a $SRG(2^n, \Omega_f, e, d)$ with

$$Spec(G_f) = (|\Omega_f|^0, \frac{1}{2}(e-d + \sqrt{(e-d)^2 - 4d|\Omega_f|})^0, \frac{1}{2}(e-d - \sqrt{(e-d)^2 - 4d|\Omega_f|})^0)$$

Theorem if $f$ is a bent function with connected $G_f$, then $G_f$ is a strongly regular graph $SRG(v, k, l, d)$ with

$$(v, k, l) = \begin{cases} (2^n, 2^{n-1} + 2^{\frac{n-1}{2}}, 2^{n-2} + 2^{\frac{n-1}{2}}, 2^{n-2} + 2^{\frac{n-1}{2}}) & \text{or} \\ (2^n, 2^{n-1} - 2^{\frac{n-1}{2}}, 2^{n-2} - 2^{\frac{n-1}{2}}, 2^{n-2} - 2^{\frac{n-1}{2}}) \end{cases}$$
\[ \text{Spec}(G) = \{(2^{n-1} + 2^{\frac{n-1}{2}})^{(1)}, (2^{\frac{n-1}{2}})^{(2^{n-1} - 2^{\frac{n-1}{2}} - 1)}, (-2^{\frac{n-1}{2}})^{(2^{n-1} + 2^{\frac{n-1}{2}})} \} \]

\[ \text{Spec}(G) = \{(2^{n-1} - 2^{\frac{n-1}{2}})^{(1)}, (2^{\frac{n-1}{2}})^{(2^{n-1} - 2^{\frac{n-1}{2}} + 1)}, (-2^{\frac{n-1}{2}})^{(2^{n-1} + 2^{\frac{n-1}{2}})} \}. \]

3. Strongly Regular Graphs \( \text{SRG}(n,k,\lambda,\bar{\lambda}) \)

The Friendship theorem shows that a connected graph with a unique common neighbor for any pairs of distinct vertices has a vertex adjacent to its all other vertices, and \( K_3 \) is the unique such regular graph. We now consider those connected \( k \)-regular graphs such that any two distinct vertices has a constant number of \( \lambda \) common neighbors, they are strongly regular graphs \( \text{SRG}(n, k, \lambda, \bar{\lambda}) \). When \( \lambda = 1 \), then \( G = K_3 \) as just mentioned. The symplectic graphs \( \text{Sp}(2m) \) offer a family of such strongly regular graphs with parameters \( (2^{2m-1}, 2^{2m-1} - 1, 2^{2m-2}, 2^{2m-2}) \) for positive integers \( m \), note that \( K_3 \) is the symplectic graph \( \text{Sp}(2) \). The Cayley graphs associated with bent functions provide another family of such graphs.

**Theorem:** Suppose there exists a \( \text{SRG}(n, k, \lambda, \bar{\lambda}) \) with \( \lambda > 1 \), and with distinct eigenvalues \( k, \theta, \tau \) are integers with multiplicities

\[ m_\theta = \frac{1}{2}((n-1) - \frac{k}{\sqrt{k-\lambda}}), \text{ and } \quad m_\tau = \frac{1}{2}((n-1) + \frac{k}{\sqrt{k-\lambda}}). \]

1. \( \theta \mid \lambda \) and \( (n, k) = \left(\frac{(\theta^2 + \theta + \lambda)(\theta^2 - \theta + \lambda)}{\lambda}, \theta^2 + \lambda\right) \).

Proof: (1). Available in monographs, omitted. (2) Let \( t = \frac{k}{\sqrt{k-\lambda}} \), which is a positive integer by (1). Hence \( k = \frac{t^2 \pm t\sqrt{t^2 - 4\lambda}}{2} \), both \( t \) and \( b = \sqrt{t^2 - 4\lambda} \) are of the same parity; since \( t^2 - 4\lambda = b^2 \), it follows that \( 4\lambda = (t + b)(t - b) \), both

\[ t + b = \frac{k}{\sqrt{k-\lambda}} + \sqrt{\left(\frac{k}{\sqrt{k-\lambda}}\right)^2 - 4\lambda}, \quad \text{and} \quad t - b = \frac{k}{\sqrt{k-\lambda}} - \sqrt{\left(\frac{k}{\sqrt{k-\lambda}}\right)^2 - 4\lambda} \]

must be even. Let \( t + b = 2h_1 \) and \( t - b = 2h_2 \) for some positive integers \( h_1 > h_2 \), hence \( \lambda = h_1h_2 \), then \( t = h_1 + h_2, b = h_1 - h_2, \) and \( k \) is either \( h_1(h_1 + h_2) \) or \( h_2(h_1 + h_2) \). Note that \( \theta = \sqrt{k-\lambda} \) is either \( h_1 \) (in case \( k = h_1(h_1 + h_2) \)) or \( h_2 \) (in case \( k = h_2(h_1 + h_2) \)), hence \( \theta \mid \lambda \). It follows that \( n = \left(\frac{(\theta^2 + \theta + \lambda)(\theta^2 - \theta + \lambda)}{\lambda}\right) \) in either case as required. Q.E.D.

The above lemma paves a way for studying possible feasible parameters \((v, k, \lambda, \bar{\lambda})\) for a given \( \lambda \) with a pair \((h_1, h_2) = (\theta, \lambda / \theta) \) or \((\lambda / \theta, \theta)\). The trivial decomposition of \( \lambda = 1 \cdot \bar{\lambda} \)
with \((h_1, h_2) = (\lambda, 1)\) leads to 
\((\nu, k, \lambda, \lambda) = (\lambda^2(\lambda + 2), \lambda(\lambda + 1), \lambda, \lambda)\) or 
\((\lambda + 2, \lambda + 1, \lambda, \lambda)\).

The other extremal cases with \(h_1, h_2\) close to \(\sqrt{\lambda}\) are considered for \(\lambda = 2^m\), and 
\(2^m(2^m + 1)\) respectively.

If \(\lambda = 2^m\) with \((h_1, h_2) = (2^m, 2^m)\), then 
\((\nu, k, \lambda, \lambda) = (2^{2m+2} - 1, 2^{2m+1}, 2^m, 2^m)\) which is 
identical with those of the symplectic graphs.

If \(\lambda = 2^m(2^m + 1)\), then 
\((\nu, k, \lambda, \lambda) = 
(2^2(2^m + 1)^2, (2^m + 1)(2^{m+1} + 1), 2^m(2^m + 1), 2^m(2^m + 1))\) or 
\((2^m(2^{m+2}), 2^m(2^{m+1} + 1), 2^m(2^m + 1), 2^m(2^m + 1))\)
respectively with \((h_1, h_2) = (2^m + 1, 2^m)\) respectively. For the symplectic graphs \(Sp(2(m+1))\),
which is a \(SRG(2^{2m+2} - 1, 2^{2m+1}, 2^m, 2^m)\) with spectrum 
\(Spec(G) = ((2^{2m+1})^1, (2^m)^{2^{2m+1} - 2^m - 1}, (-2^m)^{2^{2m+1} + 2^m - 1})\),
some examples with small number of vertices are known already, for example:

\(SRG(3, 2, 1, 1)\) with \(Spec(G) = (2^1, 1^0, (-1)^2)\),
\(SRG(15, 8, 4, 4)\) with \(Spec(G) = (8^1, 2^5, (-2)^9)\),
\(SRG(63, 32, 16, 16)\) with \(Spec(G) = (32^1, 4^{27}, (-4)^{35})\), and
\(SRG(255, 128, 64, 64)\) with \(Spec(G) = (128^1, 8^{119}, (-8)^{135})\).

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