Discrete-time quasi-sliding-mode control for a class of nonlinear control systems

S.-D. Xu, Y.-W. Liang and S.-W. Chiou

A quasi-sliding-mode control law is proposed for a class of discrete-time nonlinear control systems concerning system stabilisation and chatter alleviation. An illustrative example is also given to demonstrate the use and benefits of the scheme.

Introduction: It is known that sliding mode control (SMC) schemes possess the benefits of fast response and low sensitivity to parameter uncertainties and disturbances [1, 2]. Therefore, they have been widely applied to control a variety of systems [1–6]. On the other hand, many control systems have been mathematically formulated as a discrete-time version because of the popularity and scale of applications of digital computers in technology and industry. Thus, the study of SMC for discrete-time systems has recently attracted considerable attention [1, 3–6]. However, owing to a finite sampling frequency characteristic in discrete-time systems, the system states can only be expected to approach the selected sliding surface and remain around it, instead of remaining on the surface when the system undergoes external disturbances. Therefore, the so-called quasi-sliding-mode (QSM) concept was introduced and discussed in discrete-time systems [1, 4, 6]. A brief summary is given here. The system states are required to monotonically approach the sliding surface until they enter the vicinity of the surface, and they then remain inside [1]. The vicinity of the sliding surface is called a quasi-sliding-mode band (QSMB). Under this QSMB definition, it is noted that the system states are not required to cross the sliding surface, as in the definition given by Gao et al. [4]. In thisLetter, we apply the QSM concept to study the stabilisation for a class of discrete-time nonlinear control systems.

Main results: Consider the discrete-time nonlinear control system:

\[ x_1[k + 1] = f_1(x_1[k], x_2[k]) \]

and

\[ x_2[k + 1] = f_2(x_1[k], x_2[k]) + G(x_1[k], x_2[k])u[k] + d[k] \]

Here, \( x_i \in \mathbb{R}^{n_i} \) and \( x_2 \in \mathbb{R}^{n_2} \) are state vectors, \( u \in \mathbb{R}^{n_u} \) denotes system input, \( d[k] \) contains possible uncertainties and/or disturbances, \( f_1, f_2 \) and \( G \) are smooth functions with appropriate dimensions, and \( f_1(0, 0) = 0 \) and \( f_2(0, 0) = 0 \). In this study, we say a vector \( a \geq 0 \) if only if \( a_i \geq 0 \) for each component \( a_i \) of \( a \). Below we impose three assumptions for System (1)–(2):

Assumption 1: \( G(\cdot) \) is a non-singular matrix for all state vectors.

Assumption 2: There exists a function \( \phi(x_2[k], \phi(0) = 0, \) such that the origin of the reduced order system is \( x_1[k+1] = f_1(x_1[k]) \) is asymptotically stable (AS).

Assumption 3: \( d_{1}[k] \leq d[k] \leq d_{2}[k] \), where \( d_{1} \) and \( d_{2} \) are two constant vectors.

Owing to the aforementioned merits of SMC designs, we apply the QSM approach to the controller design. From Assumption 2, we select the sliding surface to be:

\[ s[k] = x_2[k] - \phi(x_2[k]) = 0 \]

Clearly, the origin of the reduced order dynamics (i.e. set \( s[k] = 0 \)) is AS under Assumption 2. To derive a suitable \( \phi \) for a sliding surface, several approaches have been proposed. For instance, Kalman and Bertram [7] used the second method of Lyapunov, while Zheng et al. [5] adopted the linear matrix inequalities (LMI) technique.

To organise an appropriate QSM controller, we adopt the approach of Bartosziewicz [1] which was developed for linear systems. Suppose that \( s_1[k] \) has a desired sliding variable trajectory. One candidate of \( s_1[k] \) has the following form [1]:

\[ s_1[k] = k^* - k, \quad s_1[k] = 0, \quad s_1[k] = k^* \]

where \( k^* \) is a positive integer selected by the designer to make the system states reach the sliding surface in \( k^* \) steps; the choice depends on the desired control performance. The maximum control magnitude a system control may provide. Once \( s_1[k] \) is determined, we choose the QSM controller to be:

\[ u[k] = G(x_1[k], x_2[k])^{-1}[\phi_1(x_1[k], x_2[k])] - f_2(x_1[k], x_2[k]) - d_1 + s_1[k] + d_{1}[k] \]

where \( d_{1}[k] = 1/(d_{1} + d_{2}) \). Substituting (1), (2) and (3) into (4), we have \( s_1[k] = s_1[k + 1] \leq ||d[k] - d_{1}[k]|| \leq d_{1}, \) where \( d_{1} \) is \( (1/2)(d_{1} + d_{2}) \) and \( || \cdot || \) is a vector norm. Thus, the system states will approach the sliding surface and remain around there after \( k^* \) steps. The constant \( d_{1} \) is the width of the QSMB. It is worth noting that the asymptotic stability performance can be assured if \( d_{1}[k] \leq 0 \).

If the change rate of \( d[k] \) is limited by the relation \( d[k+1] - d[k]|| \leq d_{1}, \) where \( d_{1} \) is a known constant and \( d_{1} \leq d_{1}, \) then the control given by (5) may be modified as:

\[ u[k] = G(x_1[k], x_2[k])^{-1}[\phi_1(x_1[k], x_2[k])] - f_2(x_1[k], x_2[k]) - d_1 + s_1[k] + \sum_{i=0}^{\infty}(s_1[i] - s_1[i+1]) \]

We now show, by mathematical induction, that \( s_1[k + 1] - s_1[k] \leq d_{1} \) for all \( k \). From (1)–(3), the modified law and the fact \( s_1[0] = s_1[0], \) we have \( s_1[1] = s_1[1] = d_1[0] - d_1 \) and \( s_1[2] - s_1[2] = d_1[1] - d_1 - (s_1[1] - s_1[1]) = d_1[1] - d_1[0]. \) Suppose that \( s_1[k + 1] - s_1[k] = d_1[1] - d_1[1] - d_1[2] \) for all \( 2 < k \leq k, \) then \( s_1[k + 1] - s_1[k] \leq d_1[1] - d_1[1] - d_1[2] \) for all \( 2 < k \leq k. \) Thus, \( s_1[k + 1] - s_1[k] \leq d_1[1] \). Therefore, since \( d_{1} < d_{1} \), it implies that this QSMB width is smaller than the former one.

Example: Consider a trailer-truck kinematic model [8]:

\[ x_1[k + 1] = x_1[k] + \frac{\eta T}{L} \sin(x_1[k]) + \frac{\eta T}{L} \tan(u[k]) + d[k] \]

\[ x_2[k + 1] = x_2[k] + \frac{\eta T}{L} \sin(x_1[k]) \]

\[ x_2[k + 1] = x_2[k] + \frac{\eta T}{L} \cos(x_1[k]) \sin \left( \frac{x_1[k] + x_2[k] + 1}{2} \right) \]

Here, the three states \( x_1[k], x_2[k] \) and \( x_1[k] \) are the angle difference between the trailer and the possible disturbances, \( L \) is the length of the trailer, \( l \) denotes the sampling period, and \( \eta \) is the constant speed of the backward movement. The geometry of the system can be found in Takana et al. [8]. By letting \( x_1[k] = (x_1[k], x_1[k]) \), \( x_2[k] = x_2[k] \) and \( u[k] = \tan(u[k]) \), System (7)–(9) can be put into the form of (1)–(2). The control objective is to realise the backward movement for the trailer-truck along the horizontal line \( x_1 = 0 \) without any forward movement; that is, to realise \( x_0 = x_0 \rightarrow x_0 = x_0 \rightarrow x_0 \rightarrow x_0 \).

The parameters and initial states in this example are selected as follows: \( L = 0.135 \), \( l = 0.087 \), \( \eta = 0.36 \), \( m_s = 0.5 \), \( x_1[0] = (1.571, 1), \) and \( x_0[0] = 0 \). The function \( \phi \) given in (3) is set to linear in the form of \( \phi(x_2[k]) = \frac{e^2}{2} \). Then \( \alpha = 1.3325 \), \( -3.97 \) is the vector such that the linearisation of the reduced-order model given in Assumption 2 has eigenvalues at 0.7 and 0.75. The desired sliding variable trajectory \( s_1[k] \) is taken in the form of (4) with \( k^* = 10 \). To demonstrate the robustness of the proposed schemes, we choose a slow varying disturbance, \( d[k] = 0.1 \sin(0.1k). \)

Numerical simulations are given in Figs. 1 and 2. Among these, three different control laws are adopted. Two of them are the QSM controller (5) (labelled by QSMC1) and the modified QSM controller (6) (labelled by QSMC2), while the other is an existing fuzzy scheme [8] (labelled by fuzzy). It is observed from Figs. 1a–c that the states by the three schemes exhibit oscillation because of the effect of disturbance; however, the amplitudes of the oscillation by the two QSM schemes are much smaller than those of the fuzzy design. This demonstrates the robustness characteristics of the QSM design. Fig. 2 shows the time response of the sliding variables and their magnified scale by the two QSM schemes. Both of the sliding variables are seen to approach the selected sliding surface within 10 steps, as desired. By direct
inspection, the QSMB widths of the QSMC1 and the QSMC2 schemes are $\delta_0 = 0.2$ and $\Delta_x = |x_k - x_{k-1}| \approx 0.01$, respectively, which agree with the theoretical results. In addition, the proposed QSM schemes do not require the system states to cross the sliding surface at each step (i.e. each sampling instant) when the states are within the QSMB. Therefore, the chattering phenomenon can be greatly alleviated when comparing to the QSM definition given by Gao et al. [4]. By direct calculation based on Fig. 1d, we have $(||u||_2)_{QSMC2} = 0.3946 < (||u||_2)_{QSMC1} = 0.3981 < (||u||_2)_{Fuzzy} = 0.4771$ and $(||u||_2)_{QSMC1} = 2.0157 < (||u||_2)_{QSMC2} = 2.0424 < (||u||_2)_{max} = 2.9723$, where $||u|| := \max_k |u[k]|$ and $||u||_2 := \sum_k u[k]$. From this example, it is concluded that the proposed QSM schemes are not only more robust than the existing fuzzy controller, but are also able to alleviate chatter without creating an extra control burden.

**Fig. 1 Time histories**

a–c Time history of system states
d Time history of control by the three schemes

**Fig. 2 Time histories of sliding variables**

a Time history of sliding variables
b Time history of sliding variables on magnified scale by the two QSM schemes

Conclusions: A quasi-sliding-mode control scheme for system stabilisation is proposed for a class of discrete-time nonlinear control systems. It has been shown that the scheme not only achieves the stabilisation performance, but it also alleviates chatter without creating an extra control burden.

© The Institution of Engineering and Technology 2008
16 April 2008

Electronics Letters online no: 20081070
doi: 10.1049/el:20081070

S.-D. Xu, Y.-W. Liang and S.-W. Chiou (Department of Electrical and Control Engineering, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 30010 Taiwan, Republic of China)
E-mail: ywliang@ee.nctu.edu.tw

**References**