Two Doyen–Wilson theorems for maximum packings with triples

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Abstract

In this paper we complete the work begun by Mendelsohn and Rosa and by Hartman, finding necessary and sufficient conditions for a maximum packing with triples of order \( m \) \( \text{MPT}(m) \) to be embedded in an \( \text{MPT}(n) \). We also characterize when it is possible to embed an \( \text{MPT}(m) \) with leave \( L_1 \) in an \( \text{MPT}(n) \) with leave \( L_2 \) in such a way that \( L_1 \subseteq L_2 \).

1. Introduction

A packing with triples of order \( n \) (or a partial Steiner triple system) is an ordered triple \( (S,T,L) \) where \( T \) is a set of edge-disjoint copies of \( K_3 \) (triples) in \( K_n \) with vertex set \( S \), and \( L \) is the set of edges in \( K_n \) belonging to no triple in \( T \). \( L \) is known as the leave of the packing. A maximum packing with triples (or simply a maximum packing) of order \( n \), denoted by \( \text{MPT}(n) \), is a packing with triples \( (S,T,L) \) of order \( n \) such that for any other packing with triples \( (S',T',L') \) of order \( n \) such that for any other packing with triples \( (S',T',L') \) of order \( n \), \( |T| \geq |T'| \) (or, if you prefer, \( |L| \leq |L'| \)). A maximum packing \( (S,T,L) \) of order \( n \) with \( L = \emptyset \) is, of course, a Steiner triple system, \( \text{STS}(n) \). It has been known since 1847 [6] that an \( \text{STS}(n) \) exists if and only if \( n \equiv 1 \) or \( 3 \) (mod 6). It is also well known that for the other possible values of \( n \), the subgraph induced by the leave is: a 1-factor if \( n \equiv 0 \) or \( 2 \) (mod 6); a tripole (that is, a spanning subgraph of \( K_n \) in which one vertex has degree 3 and the rest have degree 1) if \( n \equiv 4 \) (mod 6); and a cycle of length 4 if \( n \equiv 5 \) (mod 6). (The 3 adjacent edges in a tripole are called the head of the tripole.)
Table 1
Leaves of maximum packings

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<th>0 (mod 6)</th>
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The packing $(S_1, T_1, L_1)$ of order $m$ is said to be embedded in the packing $(S_2, T_2, L_2)$ if $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$. In 1973, Doyen and Wilson [2] set the standard for embedding problems by proving the following result.

**Theorem 1.1** (Doyen and Wilson [2]). Let $m, n \equiv 1$ or $3 \pmod{6}$. Any STS($m$) can be embedded in a STS($n$) if $n \geq 2m + 1$.

This lower bound on $n$ is the best possible. Over the past 25 years much effort has been focussed on proving a similar theorem for embedding any partial STS($m$) in a STS($n$). The best result to date is that a partial STS($m$) can always be embedded in a STS($n$) for all $n \geq 4m + 1$ and $n \equiv 1$ or $3 \pmod{6}$ [1]. The best possible result would be $n \geq 2m + 1$ and $n \equiv 1$ or $3 \pmod{6}$.

In 1983, Mendelsohn and Rosa [7] considered the following generalization of Theorem 1.1. For which values of $m$ and $n$ can any maximum packing of order $m$ be embedded in a maximum packing of order $n$? It is easy to see that the following are necessary conditions.

**Lemma 1.2.** Let $n > m$. Suppose that any MPT($m$) can be embedded in a MPT($n$). Then

1. if $m = 6$ then $n = 7$ or $n \geq 10$,
2. if $m > 6$ and $m$ is even then $n = m + 1$ or $n \geq 2m$, and
3. if $m > 6$ and $m$ is odd then $n \geq 2m$.

**Remark.** There is no restriction on $n$ if $m \leq 5$.

Mendelsohn and Rosa obtained a partial answer to this problem with the following theorem.

**Theorem 1.3** (Mendelsohn and Rosa [7]). Let $s, t \in \{0, 1, 2, 3, 4, 5\}$ such that $s \in \{4, 5\}$ if and only if $t \in \{4, 5\}$. Then the necessary conditions in Lemma 1.2 for the
embedding of an MPT(m) in an MPT(n) are sufficient if

1. \( m \leq 5 \), or
2. \( m = 6 \) and \( n \in \{10, 11\} \), or
3. \( m \) is even and \( n = m + 1 \), or
4. \( m \equiv s \pmod{6} \) and \( n \equiv t \pmod{6} \).

Furthermore, the smallest possible embedding when \( n \geq 2m \) has been found in many cases by Mendelsohn and Rosa [7], by Hartman et al. [5], and in the remaining cases by Hartman [4], as the following theorem states.

**Theorem 1.4** (Hartman [4], Hartman et al. [5] and Mendelsohn and Rosa [7]). Let \( 2m \leq n \leq 2m + 5 \). Any MPT(m) can be embedded in an MPT(n).

In this paper, we finish the proof of this problem, showing that the necessary conditions of Lemma 1.2 are sufficient for the embedding of an MPT(m) into an MPT(n). To do so, we need to prove the following.

**Theorem 1.5.** Suppose that \( s \in \{0, 1, 2, 3\} \) and \( t \in \{4, 5\} \) or \( s \in \{4, 5\} \) and \( t \in \{0, 1, 2, 3\} \). Suppose also that \( m \equiv s \pmod{6} \), \( n \equiv t \pmod{6} \), and \( n > 2m \). Then any MPT(m) can be embedded in an MPT(n).

This result is proved in Sections 2 and 4. These results are all collected into one result in Section 5.

### 2. The case \( s \in \{4, 5\} \) and \( t \in \{0, 1, 2, 3\} \)

For the purposes of this paper, a *difference triple* is a 3-element set \( \{x, y, z\} \) of distinct positive integers such that \( x + y = z \).

If \( \{x, y\} \) is an edge in \( K_n \) with vertex set \( \mathbb{Z}_n \), then \( \{x, y\} \) is said to have *length* \( \ell(x, y) = \min\{y - x (\text{mod } n), x - y (\text{mod } n)\} \); so \( 1 \leq \ell(x, y) \leq n/2 \). For any subset \( L \subseteq \mathbb{Z}_{[n/2]} \), let \( G_n(L) \) be the graph with vertex set \( \mathbb{Z}_n \) and edge set \( \{\{x, y\} | \ell(x, y) \in L\} \). The following is a special case of an extremely useful lemma of Stern and Lenz.

**Lemma 2.1** (Stern and Lenz [8]). If \( n/2 \in L \) then there exists a 1-factorization of \( G_n(L) \).

The following lemma was essentially proved by Stern and Lenz [8] see also [3, Lemma 6.1]; the additional property that \( D \) can be defined so that it contains \( \{1, 2, 3\} \) for the small values of \( L \) was proved by Fu et al. (see [3, Lemmas 6.5 and 6.7]).

**Lemma 2.2.** (i) For all \( h \geq 2 \), the set \( \{1, 2, \ldots, 3h\} \setminus \{a, b, c\} \) for some \( \{a, b, c\} \subseteq \{4, 5, \ldots, 3h\} \) can be partitioned by a set \( D \) of difference triples, and if \( h \leq 10 \) then \( D \) can be defined so that \( \{1, 2, 3\} \in D \).
(ii) For all $h \geq 3$, the set $\{1,2,\ldots,3h+1\} \setminus \{a,b,c,d\}$ for some $\{a,b,c,d\} \subseteq \{5,6,\ldots,3h+1\}$ can be partitioned by a set $D$ of difference triples, and if $h \leq 13$ then $D$ can be defined so that $\{1,2,3\} \in D$.

The next two lemmas are the crucial ingredients used to prove Theorem 1.5.

Lemma 2.3. Let $h \geq 2$. The edges in $G_{6h+2}(\{1,2,3\})$ can be partitioned into 4 matchings, each of which saturates all vertices except for the vertices 3 and 10, together with a set $T$ of $2h+2$ triples.

Proof. The proof consists of defining $T$ and two cycles $c_1$ and $c_2$, each of which passes through each vertex except for 3 and 10. Then clearly $c_1$ and $c_2$ can each be partitioned into two matchings as required.

Let
\[
T = \{\{0,1,3\}, \{2,3,4\}, \{3,5,6\}, \{7,8,10\}, \{9,10,11\}, \{10,12,13\}\}
\cup \{\{3i+1,3i+2,3i+4\}, 4 \leq i \leq 2h-1\}.
\]

Let $c_1 = (x_0, x_1, \ldots, x_{6h-1})$ where $(x_0, x_1, \ldots, x_{11}) = (0,2,5,4,6,7,9,8,11,13,15,12)$, and for $2 \leq i \leq h-1$ define $(x_{6i}, x_{6i+1}, \ldots, x_{6i+5}) = (6i+2,6i+5,6i+4,6i+7,6i+9,6i+6)$, reducing sums modulo $6h+2$. Let $c_2 = (y_0, y_1, \ldots, y_{6h-1})$ where $(y_0, y_1, \ldots, y_{11}) = (0,6h+1,2,1,4,7,5,8,6,9,12,11)$, and for $2 \leq i \leq h-1$ define $(y_{6i}, y_{6i+1}, \ldots, y_{6i+5}) = (6i+2,6i+5,6i+4,6i+7,6i+6,6i+3,6i+5)$.

Lemma 2.4. Let $h \geq 2$. The edges in $G_{6h+4}(\{1,2,3,4\})$ can be partitioned into 4 matchings, each of which saturates all of the vertices except for vertices 6 and 9, together with a set $T$ of $4h+4$ triples.

Proof. Let
\[
T = \{\{0,2,4\}, \{1,3,5\}, \{2,3,6\}, \{4,6,8\}, \{5,6,9\}, \{6,7,10\}, \{7,8,9\}, \{9,10,11\}, \{9,12,13\}, \{11,12,15\}, \{13,14,15\}, \{14,16,17\}\}
\cup \{\{6i+4,6i+6,6i+7\}, \{6i+5,6i+6,6i+9\}, \{6i+7,6i+8,6i+9\}, \{6i+8,6i+10,6i+11\}\} 2 \leq i \leq h-1\}.
\]

Let $c_1$ be the $(6h+2)$-cycle $(x_0, x_1, \ldots, x_{6h+1})$, where $(x_0, x_1, \ldots, x_{13}) = (3,4,5,7,11,8,10,12,14,18,15,17,13,16)$ and for $3 \leq i \leq h$ define $(x_{6i+2}, x_{6i+3}, \ldots, x_{6i+7}) = (6i+2,6i+6,6i+3,6i+5,6i+1,6i+4)$, reducing sums modulo $6h+4$. Let $c_2$ be the $(6h-2)$-cycle $(y_0, y_1, \ldots, y_{6h-3})$, where $(y_0, y_1, \ldots, y_9) = (3,7,4,1,2,5,8,12,16,15)$ and for $3 \leq i \leq h$ define $(y_{6i-8}, y_{6i-7}, \ldots, y_{6i-3}) = (6i+1,6i-1,6i+2,6i,6i+4,6i+3)$. Finally, let $c_3$ be the 4-cycle $(10,13,11,14)$. Then $c_2$ and $c_3$ provide 2 matchings saturating each vertex except vertices 6 and 9, as does $c_1$. □
Lemma 2.5. Let \( n = 6h + 2 \). For any \( t \) with \( 1 \leq t < h \), where if \( t \leq 2 \) then \( h \leq 10 \), there exists a partition of the edges of \( K_n \) into

(i) 4 matchings that each saturates all the vertices in \( K_n \) except for \( u, v \in V(K_n) \);
(ii) \( 6t + 1 \) 1-factors of \( K_n \); and
(iii) a set \( T \) of triples.

Proof. Let \( 1 \leq t < h \) with \( h \leq 10 \) if \( t \leq 2 \). By Lemma 2.2(i), \( \{1, 2, \ldots, 3h\} \setminus \{a, b, c\} \) with \( \{1, 2, 3\} \cap \{a, b, c\} = \emptyset \) can be partitioned into a set \( D \) of \( h - 1 \) difference triples. Let \( D_1 \) be the set of difference triples in \( D \) that contain 1, 2 and 3 and let \( \alpha = |D_1| \); then clearly \( 1 \leq \alpha \leq 3 \), and by Lemma 2.2(i) if \( t \leq 2 \) then \( D_1 = \{\{1, 2, 3\}\} \), so \( \alpha = 1 \). Let \( D_2 \) be a set of \( t - \alpha \) difference triples in \( D \setminus D_1 \) (clearly \( 0 \leq t - \alpha \leq |D \setminus D_1| \)). Let

\[
L = \{a, b, c, 3h + 1\} \cup \{x \mid x \text{ is in a difference triple in } D_1 \cup D_2 \}\setminus \{1, 2, 3\}.
\]

Then \( |L| = 4 + 3\alpha + 3(t - \alpha) - 3 = 3t + 1 \). Since \( 3h + 1 \in L \), by Lemma 2.1 the edges in \( G_{6h+2}(L) \) can be partitioned into \((6t + 1)\) 1-factors. By Lemma 2.3 the edges in \( G_{6h+2}(\{1, 2, 3\}) \) can be partitioned into 4 matchings that each saturates all the vertices in \( K_n \) except for two, together with a set \( T_1 \) of triples. Finally, the edges in \( K_n \) of lengths in the difference triples in \( D \setminus (D_1 \cup D_2) \) are partitioned by the triples in

\[
T_2 = \{\{i, x + i, x + y + i\} \mid \{x, y, x + y\} \in D \setminus (D_1 \cup D_2), i \in \mathbb{Z}_n\}.
\]

So setting \( T = T_1 \cup T_2 \) provides the required partition. \( \Box \)

Lemma 2.6. Let \( n = 6h + 4 \). For any \( t \) with \( 1 \leq t < h \), where if \( t \leq 3 \) then \( h \leq 13 \), there exists a partition of the edges of \( K_n \) into

(i) 4 matchings that each saturates all the vertices in \( K_n \) except for \( u, v \in V(K_n) \);
(ii) \( 6t + 1 \) 1-factors; and
(iii) a set \( T \) of triples.

Proof. Let \( 1 \leq t < h \) with \( h \leq 13 \) if \( t \leq 3 \). If \( h = 2 \) (so \( t = 1 \)), let \( L = \{5, 6, 7, 8\} \) and \( D = \emptyset \).

If \( h \geq 3 \) define \( L, D, D_1 \) and \( D_2 \) as follows. By Lemma 2.2(ii) \( \{1, 2, \ldots, 3h + 1\} \setminus \{a, b, c, d\} \) with \( \{1, 2, 4\} \cap \{a, b, c, d\} = \emptyset \) can be partitioned into a set \( D \) of \( h - 1 \) difference triples. Let \( D_1 \) be the set of difference triples in \( D \) that contain 1, 2, 3 and 4, and let \( \alpha = |D_1| \); then clearly \( 2 \leq \alpha \leq 4 \), and by Lemma 2.2(ii) if \( t \leq 3 \) then \( \alpha = 2 \) (since \( \{1, 2, 3\} \subseteq D \)). Let \( D_2 \) be a set of \( t - \alpha \) difference triples in \( D \setminus D_1 \) (clearly \( 0 \leq t - \alpha \leq |D \setminus D_1| \)). Let

\[
L = \{a, b, c, d, 3h + 2\} \cup \{x \mid x \text{ is in a difference triple in } D_1 \cup D_2 \}\setminus \{1, 2, 3, 4\}.
\]

Then \( |L| = 5 + 3\alpha + 3(t - \alpha) - 4 = 3t + 1 \).

In any case, since \( 3h + 2 \in L \), by Lemma 2.1, the edges in \( G_{6h+4}(L) \) can be partitioned into \((6t + 1)\) 1-factors. By Lemma 2.4 the edges in \( G_{6h+4}\{1, 2, 3, 4\} \) can be partitioned into 4 matchings that each saturates all the vertices in \( K_n \) except for two, together
with a set $T_1$ of triples. Finally, the edges in $K_n$ of lengths in the difference triples in $D(D_1 \cup D_2)$ are partitioned by the triples in
\[ T_2 = \{ \{i, x + i, x + y + i\} \mid \{x, y, x + y\} \in D(D_1 \cup D_2), \ i \in \mathbb{Z}_n \}. \]
So setting $T = T_1 \cup T_2$ provides the required partition. \(\square\)

**Proposition 2.7.** Let $m \equiv 5 \pmod{6}$, $n \equiv 1$ or $3 \pmod{6}$, and $n > 2m$. Any maximum packing of order $m$ can be embedded in a $STS(n)$.

**Proof.** Let $m = 6t + 5$ and $n - m \in \{6h + 2, 6h + 4\}$. Let $(S_1, T_1, L_1)$ be a maximum packing of order $m$, and let $S_2$ be a set of size $n - m$ with $S_1 \cap S_2 = \emptyset$. By Theorem 1.3 we can assume that $m \geq 11$. So, since $n > 2m$, we have that $1 \leq t < h$.

Suppose first that if $t \leq 2$ and $n - m = 6h + 2$ then $h \leq 10$, and if $t \leq 3$ and $n - m = 6h + 4$ then $h \leq 13$. Then by Lemmas 2.5 and 2.6, there exists a partition of the edges of $K_{n-m}$ with vertex set $S_2$ into $4$ matchings $M_1, M_2, M_3$ and $M_4$ that each saturates all vertices in $S_2$ except for $u, v \in S_2$; $t + 1$ $1$-factors $F_1, F_2, \ldots, F_{t+1}$ of $K_{n-m}$; and a set $T$ of triples. Let $L_1 = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_1, a_4\}\}$ and let $\phi: S_1 \cup \{a_1, a_2, a_3, a_4\} \to \{1, 2, \ldots, 6t + 1\}$ be a $1$-$1$ mapping. Define
\[ T_2 = T_1 \cup \{\{a_1, a_2, u\}, \{a_3, a_4, u\}, \{a_1, a_4, v\}, \{a_2, a_3, v\}\} \]
\[ \cup \{\{a_i, x, y\} \mid 1 \leq i \leq 4, \{x, y\} \in M_i\} \cup \{\{s, x, y\} \mid s \in S_1, \{x, y\} \in F_{\phi(s)}\}. \]
Then clearly $(S_1 \cup S_2, T_2)$ is an $STS(n)$, and $(S_1, T_1, L_1)$ is embedded in this $STS(n)$.

Now suppose that either $1 \leq t \leq 2$, $n - m = 6h + 2$ and $h \geq 11$ (so $n = 6t + 5 + 6h + 2 \geq 79$), or $1 \leq t \leq 3$, $n - m = 6h + 4$ and $h \geq 14$ (so $n \geq 99$). If $n - m = 6h + 2$ then the previous case shows that $(S_1, T_1, L_1)$ can be embedded in a $STS(37)$ (that is, $37 = 6t + 5 + 6h' + 2$, where $h' = 4$ or $3$ if $t = 1$ or $2$, respectively), and by Theorem 1.1 this $STS(37)$ can be embedded in a $STS(n)$ since $n \geq 75$. If $n - m = 6h + 4$ then we have shown that $(S_1, T_1, L_1)$ can be embedded in a $STS(49)$, which by Theorem 1.1 can be embedded in a $STS(n)$ since $n \geq 99$.

Thus, in any case $(S_1, T_1, L_1)$ has been embedded in a $STS(n)$. \(\square\)

**Proposition 2.8.** Let $m \equiv 5 \pmod{6}$ and $n \equiv 0$ or $2 \pmod{6}$, or $m \equiv 4 \pmod{6}$ and $n \equiv 0, 1, 2$ or $3 \pmod{6}$. Let $n > 2m$. Then any maximum packing of order $m$ can be embedded in a maximum packing of order $n$.

**Proof.** Let $(S_1, T_1, L_1)$ be a maximum packing of order $m$.

If $m \equiv 5 \pmod{6}$ then by Proposition 2.7, $(S_1, T_1, L_1)$ can be embedded in a $STS(n+1)$ $(S_2, T_2)$. For any $s \in S_2 \setminus S_1$, $(S_2 \setminus \{s\}, \{t \mid t \in T_2 \text{ and } s \notin t\})$ is a maximum packing of order $n$ that contains $(S_1, T_1, L_1)$.

If $m \equiv 4 \pmod{6}$ then $L_1$ is a tripole, say $L_1 = \{\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}\} \cup \{\{s_{2i-1}, s_{2i}\} \mid 3 \leq i \leq m/2\}$. Then $(S_1, T_1, L_1)$ is embedded in the maximum packing $(S_1 \cup \{s\}, T_2, L_2)$ of order $m + 1$, where $T_2 = \{\{s, s_1, s_4\}\} \cup \{\{s, s_{2i-1}, s_{2i}\} \mid 3 \leq i \leq m/2\} \cup T_1$ and $L_2 =$
\({\{s, s_2\}, \{s_2, s_1\}, \{s_1, s_3\}, \{s, s_3\}\}}\). Since \(m + 1 \equiv 5 \pmod{6}\) and since \(n > 2m\), by Proposition 2.7, \((S_1 \cup \{s\}, T_2, L_2)\) can be embedded in: a STS\((n)\) \((S_3, T_3)\) if \(n \equiv 1\) or \(3 \pmod{6}\), thus providing the required embedding; and in a STS\((n + 1)\) \((S_3, T_3)\) if \(n \equiv 0\) or \(2 \pmod{6}\), in which case \((S_3 \setminus \{s\}, \{t \mid t \in T_3 \text{ and } s \notin t\}, \{\{a, b\} \mid \{s, a, b\} \in T_3\})\) provides the required embedding. 

3. Another embedding problem

There is a second generalization of Theorem 1.1 that makes sense to consider, and that is to ensure that the leave of the MPT\((m)\) is preserved during the embedding. More specifically, the second problem is to find the integers \(m\) and \(n\) for which any MPT\((m)\) \((S_1, T_1, L_1)\) can be embedded in an MPT\((n)\) \((S_2, T_2, L_2)\) so that \(L_1 \subseteq L_2\). This extra requirement that \(L_1 \subseteq L_2\) restricts more severely the integers \(n\) for which such an embedding is possible. It is easy to see that the following conditions are necessary.

**Lemma 3.1.** Let \(n > m\). Suppose that any MPT\((m)\) \((S_1, T_1, L_1)\) can be embedded in an MPT\((n)\) \((S_2, T_2, L_2)\) with \(L_1 \subseteq L_2\). Then

1. if \(m \equiv 0\) or \(2 \pmod{6}\) then \(n\) is even,
2. if \(m \equiv 4 \pmod{6}\) then \(n \equiv 4 \pmod{6}\),
3. if \(m \equiv 5 \pmod{6}\) then \(n \equiv 5 \pmod{6}\), and
4. \(n \geq 2m\), with strict inequality if \(m \equiv 0, 2, 4\) or \(5 \pmod{6}\).

**Proof.** (1)--(3) follow directly from the requirement that \(L_1 \subseteq L_2\). The requirement that \(n \geq 2m\) follows from (1)--(3) and Lemma 1.2.

To see that if \(m = 0, 2, 4\) or \(5 \pmod{6}\) then \(n \neq 2m\), consider the following. If \(n = 2m\) then (1)--(3) require \(m\) to be even, so the leave of \((S_1, T_1, L_1)\) is a spanning graph. Therefore, all the edges joining vertices in \(S_1\) to vertices in \(S_2 \setminus S_1\) (except for at most two such edges that could occur in the head of the tripole when \(n \equiv 4 \pmod{6}\)) must occur in triples that contain one vertex in \(S_1\) and two vertices in \(S_2 \setminus S_1\). It is easy to check that there are not enough edges joining vertices in \(S_2 \setminus S_1\) for this to be possible. Hence, in these cases \(n > 2m\). 

It turns out that in many cases, the embeddings of Mendelsohn and Rosa have this additional property.

**Theorem 3.2** (Mendelsohn and Rosa [7]). Let \(n \geq 2m\), with strict inequality if \(m \equiv 0, 2, 4\) or \(5 \pmod{6}\). Suppose that \(m, n \equiv 0\) or \(2 \pmod{6}\), or \(m \equiv n \equiv 4\) or \(5 \pmod{6}\), or \(m \equiv 1\) or \(3 \pmod{6}\) and \(n \equiv 0, 1, 2\) or \(3 \pmod{6}\). Then any MPT\((m)\) \((S_1, T_1, L_1)\) can be embedded in an MPT\((n)\) \((S_2, T_2, L_2)\) with \(L_1 \subseteq L_2\).

Therefore, to prove that the necessary conditions of Lemma 3.1 are sufficient, we need only consider the case where \(m \equiv 0\) or \(2 \pmod{6}\) and \(n \equiv 4 \pmod{6}\), and the
case where \( m \equiv 1 \text{ or } 3 \pmod{6} \) and \( n \equiv 4 \text{ or } 5 \pmod{6} \), with \( n > 2m \) in each case. In the next section we will consider these cases as we complete the proof of Theorem 1.5.

4. The case \( s \in \{0, 1, 2, 3\} \) and \( t \in \{4, 5\} \)

Two of these cases have already been settled with the following result.

**Proposition 4.1** (Fu et al. [3]). Let \( m \equiv 0 \text{ or } 2 \pmod{6} \) and \( n \equiv 4 \pmod{6} \) with \( n > 2m \). Then any MPT(m) \( (S_1, T_1, L_1) \) can be embedded in an MPT(n) \( (S_2, T_2, L_2) \) in which \( L_1 \subseteq L_2 \), and in which the edges in \( L_2 \setminus L_1 \) all join pairs of vertices that are both in \( S_2 \setminus S_1 \).

We can use Proposition 4.1 to obtain the following result.

**Proposition 4.2.** Let \( m \equiv 1 \text{ or } 3 \pmod{6} \) and \( n \equiv 4 \text{ or } 5 \pmod{6} \) with \( n > 2m \). Then any MPT(m) \( (S_1, T_1, L_1 = \emptyset) \) can be embedded in an MPT(n) \( (S_2, T_2, L_2) \) in which trivially \( L_1 \subseteq L_2 \), and in which the edges in \( L_2 \setminus L_1 \) all join vertices in \( S_2 \setminus S_1 \).

**Proof.** Suppose first that \( n = 6h + 5 \). Let \( s \in S_1 \). By Proposition 4.1 we can embed the MPT(m - 1) \( (S_1', T_1', L_1') = (S_1 \setminus \{s\}, \{t \mid t \in T_1, s \notin t\}, \{\{x, y\} \mid \{s, x, y\} \in T_1\}) \) in an MPT(n - 1) \( (S_2', T_2', L_2') \) in which \( s \notin S_2' \) and \( L_1 \subseteq L_2' \). Let \( L_2' = \{\{x, y\} \mid 1 \leq i \leq 3h\} \cup \{\{s_1, s_2\}, \{s_1, s_3\}, \{s_1, s_4\}\}, \) where by Proposition 4.1 we can assume that \( \{s_1, s_2, s_3, s_4\} \subseteq S_2 \setminus S_1 \). Then \( (S_1, T_1, L_1 = \emptyset) \) is embedded in the MPT(n) \( (S_2, T_2, L_2) = (S_2' \cup \{s\}, \{\{s, x_i, y_i\} \mid 1 \leq i \leq 3h\} \cup \{\{s, s_1, s_2\} \cup T_2\}, \{\{s, s_3\}, \{s, s_4\}, \{s_1, s_3\}, \{s_1, s_4\}\}). \) Trivially \( L_1 \subseteq L_2 \).

Also, \( (S_1, T_1, L_1) \) is embedded in the MPT(6h + 4) \( (S_2 \setminus \{s\}, \{t \mid t \in T_2, s_1 \notin t\}, \{\{x, y\} \mid \{s_1, x, y\} \in T_2\}) \), so the result is proved. \qed

**Proposition 4.3.** Let \( m \equiv 0 \text{ or } 2 \pmod{6} \) and let \( n \equiv 5 \pmod{6} \) with \( n > 2m \). Then any MPT(m) can be embedded in an MPT(n).

**Proof.** Let \( n = 6h + 5 \). Let \( (S_1, T_1, L_1) \) be an MPT(m). Then by Theorem 1.4 if \( n - 1 = 2m \) and by Proposition 4.1 otherwise, we can embed this in an MPT(n - 1), which can itself be embedded in an MPT(n) in the same way that \( (S_2', T_2', L_2') \) was embedded in \( (S_2, T_2, L_2) \) in the previous proof. \qed

5. Summary

We now collect together these results.
Theorem 5.1. Let \( n > m \). Any \( \text{MPT}(m) \) can be embedded in an \( \text{MPT}(n) \) if and only if

1. if \( m = 6 \) then \( n = 7 \) or \( n \geq 10 \),
2. if \( m > 6 \) and \( m \) is even then \( n = m + 1 \) or \( n \geq 2m \), and
3. if \( m > 6 \) and \( m \) is odd then \( n \geq 2m \).

Proof. The necessity comes from Lemma 1.2. The sufficiency follows from Theorems 1.3–1.5. Theorem 1.5 is proved by Propositions 2.7, 2.8 and 4.1–4.3.

Theorem 5.2. Let \( n > m \). Any \( \text{MPT}(m) \) \((S_1,T_1,L_1)\) can be embedded in an \( \text{MPT}(n) \) \((S_2,T_2,L_2)\) such that \( L_1 \subseteq L_2 \) if and only if

1. if \( m \equiv 0 \) or \( 2 \pmod{6} \) then \( n \) is even,
2. if \( m \equiv 4 \pmod{6} \) then \( n \equiv 4 \pmod{6} \),
3. if \( m \equiv 5 \pmod{6} \) then \( n \equiv 5 \pmod{6} \), and
4. \( n \geq 2m \), with strict inequality if \( m \equiv 0, 2, 4 \) or \( 5 \pmod{6} \).

Proof. The necessity follows from Lemma 3.1. The sufficiency is proved by Theorem 3.2 and Propositions 4.1 and 4.2.

References