國 立 交 通 大 學

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碩士論文

強正則重邊圖及其應用之研究

A Study of Strongly Regular Multigraphs with Some Applications

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指導教授: 黃大原 教授

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碩 士 論 文

A Thesis
Submitted to Department of Applied Mathematics
College of Science
National Chiao Tung University
in Partial Fulfillment of the Requirements
for the Degree of
Master
in
Applied Mathematics
June 2005
Hsinchu, Taiwan, Republic of China

中 華 民 國 九 十 四 年 六 月

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摘要

為了研究準剩餘2-設計的問題,Bose 首先於1976年提出強正則重邊圖的概念。接著 Neumaier 和 Metsch 分別在1982年及1995年也利 用強正則重邊圖的概念進一步的解決準剩餘2-設計的問題。Neumaier 在1976年的論文提到, 若強正則重邊圖的參數滿足某些條件 這個強正則重邊圖就會是唯一的$1\frac{1}{2}$-設計的共線性圖。

我們在第三節證明強正則重邊圖和強正則圖相同, 均為具有三個相異的特徵值所刻畫。針對具有三個相異特徵值的圖, 我們找出 其分別對應於強正則圖和強正則重邊圖的條件。我們在第四節, 詳細回顧Bose,Neumaier 及 Metsch 等的三篇論文, 比較他們的結 論, 及其證明所用的方法。我們在第五節給出一類源於交錯圖的強正則重邊圖的例子。同時, 我們根據 Neumaier 的定理證明他們 是唯一的$1\frac{1}{2}$-設計的共線性圖, 這項結論有助於交錯圖的幾何刻畫。

中 華 民 國 九 十 四 年 六 月

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June 2005

Abstract

The conception of strongly regular multigraph was first proposed by Bose in 1976, followed by Neumaier and Metsch in 1982 and 1995 respectively for the problem of embedding of quasi-residual 2-design. In particular, Neumaier asserted that the collinearity graph of a unique $1\frac{1}{2}$-design if it meets some constraints over its parameters.

The spectral properties of strongly regular multigraphs are studied in Section 3, we show that they can be characterized as multigraphs with exactly three distinct eigenvalues, we show further when they are strongly regular graphs in terms of their eigenvalues. For reference purpose, the results together with the arguments for the proofs of the papers of Bose, Neumaier and Metsch are summarized is Section 4. A class of strongly regular multigraphs associated with the alternating forms graphs is studied in Section 5. Under some numerical constraints, they are the collinearity graphs of uniquely determined $1\frac{1}{2}$-designs, which provide some information for the geometric characterization of the alternating forms graphs.

誌謝

剛到交大的頭一個學期，面對不同於大學的讀書習慣，讓初到交大 的我頗不習慣，也顯的恐懼。加入黃大原老師的門下之後，漸漸的習慣了這邊的 生活。老師作研究嚴謹的態度、面對問題追根究底的精神，讓我受益良多；謝謝您這兩年給予我的指導，我會謹記在心。

在此要感謝系上的黃光明老師、傅恆霖老師老師，感謝您們給予我學業上的指導。謝謝陳秋媛老師平時對我的關心，並且在我無助的時候，適時的給予我精神上的鼓勵；特別感謝翁志文老師在我學業上遇到問題的時候，給予我幫助，不厭其煩的協助我解決問題。

從東海一起來交大奮鬥的惠蘭、宏銘，大家互相鼓勵、扶持，尤其是惠蘭，我真的很高興認識妳；謝謝應數女籃的夥伴們，在課餘時間一起打球，舒緩壓力；謝謝博士班的學長們在課業上的幫助；感謝曲敏在最後幾個月的時間，耐心的聽我解釋我 的結果，並且給予我意見；感謝我的男朋友建瑋這一年來的支持與鼓勵，在我最低潮的時候，拉了我一把，使我不至於放棄我的學位，讓我堅持到最後。

最後感謝我的父母，因為您們的支持，我可以在沒有任何壓力之下，完成我的碩士學位。

非常的感謝您們。

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1 Introduction

The notion of *strongly regular multigraphs* (*SRMG*) was first introduced by R. C. Bose [2] but in a very cumbersome notation. While characterizing *quasi-residual* 2-*designs*, A. Neumaier [9] gave an equivalent definition of strongly regular multigraphs in an elegant and self-contained way with some improvements over some results; the concept "*type*" by five parameters (m, n, μ, γ, R).

Recently, Metsch [7] continued the study of embeddings of residual 2-designs within the framework of strongly regular multigraphs. However, no specific example were given in the papers mentioned above.

For a 2-(v, k, λ)-design $\pi = (X, \mathcal{B})$, it is known that every point is in $r = \frac{\lambda(v-1)}{k-1}$ blocks, and the number of blocks in $\mathcal{B}$ is $b = \frac{\lambda v(v-1)}{k(k-1)}$ and the number of points is $v = k + \frac{n(k-1)}{\lambda}$ where $n = r - \lambda$ is called the order of the design. Moreover, Fisher's inequality $b \geqslant v$ holds, with equality if and only if every pair of distinct blocks intersects in λ points. A 2-(v, k, λ)-design with $b = v$, i.e., $v = 1 + \frac{k(k-1)}{\lambda}$ is called a *symmetric design.* If B is a block of a symmetric 2-(v, k, λ)-design $\pi = (X, \mathcal{B})$, two associated designs, i.e., the *derived design* $\mathcal{B}_{der}$ and the residual design $\mathcal{B}_{res}$ with respect to the block B arise naturally. In both cases, respect to the block B arise naturally. In both cases, the blocks are the members of $\mathcal{B} - \{B\}$, and the points are the points in B for $\mathcal{B}_{der}$, and the points not in B for $\mathcal{B}_{res}$, incidences are remained the same as before. In terms of the order n, the derived design is a 2-$(n+\lambda, \lambda, \lambda-1)$-design, and the residual design is a 2-(w, n, λ)-design with $w = \frac{n(n+\lambda-1)}{\lambda}$.

The *block multigraph* of a 2-design $\pi = (X, \mathcal{B})$ is the multigraph defined over the block set $\mathcal{B}$, and two distinct vertices A, B are connected by $m_{A,B} = |A \cap B|$ edges.

Neumaier showed that the block multigraph of a 2-(v, k, λ)-design of order n is a strongly regular multigraph $SR(m, n, \mu, \gamma, R)$ with

$$SR(m, n, \mu, \gamma, R) = (k, n, k^2\lambda, k(k-1)(\lambda-1), k(n+\lambda-1)).$$

A partial converse is given in the following theorem with some constraints over its parameters:

Theorem ([9], Theorem 1.1) Every strongly regular multigraph with parameters

$$SR(m, n, \mu, \gamma, R) = (k, n, k^2\lambda, k(k-1)(\lambda-1), k(n+\lambda-1))$$

for positive integers n, $k \neq 1$, λ, and

$$n > \max\{k(k-1)\lambda^2 - (k-1)^2\lambda, \\ 2(k-1)(k^2\lambda + k\lambda - 2\lambda + 1), \tfrac{1}{2}(k^2-1)(k^2\lambda - k + 2)\}$$

is isomorphic to the block multigraph of a 2-(v, k, λ)-design with $v = k + \frac{(r-\lambda)(k-1)}{\lambda}$.

Its proof involves more general designs, namely $1\frac{1}{2}$-*designs* (called *partial geometric designs* in [2]), and *weak* $1\frac{1}{2}$-*designs* (without assuming constant block size). Note that 2-designs, dual 2-designs, transversal designs, semiregular partially balanced incomplete block designs, partial geometries, and polar spaces are examples of $1\frac{1}{2}$-designs, see Neumaier [9]. The notion of *partial geometric design* $D(r, k, t, c)$ was introduced as a generalization of a *partial geometry* (r, k, t) (with $c = 0$ above). A partial geometric design $D(r, k, t, c)$ gives rise in a natural manner to a *strongly regular multigraph* (*SRMG*) $G(D)$ whose parameters depend on r, k, t and c, as a generalization of *strongly regular graphs* (*SRG*). The block multigraphs of $1\frac{1}{2}$-designs, and dually, the point multigraphs of weak $1\frac{1}{2}$-designs still are strongly regular, and by investigating closely the properties of cliques and claws in a multigraph we obtain general characterization theorems which specialize to Theorem 1.1([9]).

The matrix techniques were used by Neumaier in order to get the relations among the five parameters, and then to derive the essential relations between strongly regular multigraph and $1\frac{1}{2}$-design.

For any two distinct blocks A, $B \neq H$, denote by $\alpha_{A,B}$ the number of points in H incident with A and B, and by $\beta_{A,B}$ the number of points not in H incident with A and B. Then $\alpha_{A,B} + \beta_{A,B} = \lambda$, in particular $\beta_{A,B} \leqslant \lambda$. Moreover, the multigraph on $\mathcal{B} - \{H\}$, with $\alpha_{A,B}$ edges between A and B, is the block multigraph of $\mathcal{B}_{der}$. Hence, the residual design satisfies the conditions given in the following theorem.

Theorem ([9], Theorem 1.2) A quasi-residual 2-(w, n, λ)-design π is embeddable if and only if the following conditions are satisfied:

(1) Any two distinct blocks A and B intersect in $\alpha_{A,B} \leqslant \lambda$ points.

(2) Let G be the multigraph defined on the blocks with $\alpha_{A,B} = \lambda - \beta_{A,B}$ edges between A and B, then

  a. G is a strongly regular multigraph $SR(m, n, \mu, \gamma, R)$ with

  $SR(m, n, \mu, \gamma, R) = (\lambda, n, \lambda^2(\lambda - 1), \lambda(\lambda - 1)(\lambda - 2), \lambda(n + \lambda - 2))$;

  b. G is isomorphic to the block multigraph of a 2-$(n + \lambda, \lambda, \lambda - 1)$-design π'.

Theorem ([9], Theorem 1.3) A quasi-residual 2-(w, n, λ)-design is embeddable if either

(1) $\lambda = 3$, and $n > 76$, or

(2) $\lambda \neq 3$, and $n > \frac{1}{2}(\lambda^2 - 1)(\lambda^3 - \lambda^2 - \lambda + 2)$.

Theorem 1.3 improved the result obtained by Bose [2] showed in 1976 that there is a polynomial function $f(\lambda)$ of degree 5 such that every quasi-residual 2-(v, k, λ) design is residual provided that $k > f(\lambda)$. It was further improved that

$f(\lambda) = (\lambda^2 - 1)(\lambda^3 - \lambda^2 - \lambda + 2)$ by Neumaier [9] in Theorem 1.3, and then

$f(\lambda) = (\frac{8}{\sqrt{3}}\lambda + \lambda + 5)\lambda^2(\lambda - 1)$ by Metsch [7].

This embedding theorem will be a consequence of more general characterization theorems for certain strongly regular multigraphs (see Theorem 2 [9] and its corollary in the introduction).

As to us, Neumaier's most important contribution is the Theorem 4.4 written in his paper. He showed that if Γ is a strongly regular multigraph $SR(m, n, \mu, \gamma, R)$ with integral $m \geqslant 2$, integral $\mu \equiv 0 \bmod m$, $\mu > 0$, and

$$n > \max\{m - 1 + \frac{(\mu + m)\gamma}{m^2}, 2(m - 1)(\mu + 1 - m) + 2\gamma, \frac{m(m-1)}{2}(\mu + 1) + m\frac{\gamma}{2} + m - 1\}$$

then Γ is the point multigraph of a unique $1\frac{1}{2}$-design, with parameters

$$(r', k', t', c') = (m, \frac{R}{m} + 1, \frac{\mu}{m}, \frac{\gamma}{m}).$$

As we know, strongly regular graph has some necessary and sufficient conditions. Two necessary and sufficient conditions on strongly regular multigraphs were given in Section 3. These papers of R. C. Bose, Neumaier, Metsch over three decades will be surveyed in Section 4, together with the technique and arguments used by them. We pay more attention for the unique theorem (Theorem 4.4 [9]) for its unique presentation of $1\frac{1}{2}$-designs. A class of specific example associated with alternating bilinear form meeting the numerical constraints will be provided in Section 5.

2 Definitions and Preliminaries

In this section, we define strongly regular multigraphs and $1\frac{1}{2}$-designs. Then we give the necessary and sufficient conditions of strongly regular multigraphs and $1\frac{1}{2}$-designs. Finally, we will give a theorem which will be useful in Section 5.

Definition 2.1 A simple graph Γ with the vertex set V and with the edge set E is called a *strongly regular graph* (SRG) with parameters (v, k, λ, μ), denoted by $SRG(v, k, \lambda, \mu)$, if

(1) $|V| = v$, and

(2) for x, $y \in V$ $|N(x) \cap N(y)| = \begin{cases} k & \text{if } x = y \\ \lambda & \text{if } x \sim y \\ \mu & \text{if } x \nsim y \end{cases}$.

A multigraph Γ contains a nonempty set V of vertices and a set E of edges. For all $x, y \in V = V(\Gamma)$, $m_{x,y} :=$ number of edges joining x and y, and define $m_{x,x} := 0$.

Definition 2.2 A multigraph Γ is called a *strongly regular multigraph* with parameters (m, n, μ, γ, R), if:

(1) $\sum_{y\in V} m_{x,y} = R$ for each $x \in V$;

(2) $\sum_{x\in V} m_{a,x}m_{b,x} = (n - 2m)m_{a,b} + m(n - m)\delta_{a,b} + \mu$, where $\delta_{a,b} = 1$ if $a = b$, otherwise, $\delta_{a,b} = 0$.

(3) $\sum_{y\in V} m_{x,y}(m_{x,y} - 1) = \gamma$ for each $x \in V$.

Here m, n, μ, γ and R are real numbers with $n > 0$.

Note that $\sum_{y\in V}(m_{x,y})^2 = \sum_{y\in V} m_{x,y} + \sum_{y\in V} m_{x,y}(m_{x,y} - 1)$ and from (1)$\sim$(3), we have

$m(n - m) + \mu = R + \gamma.$

If $\gamma = \sum_{y \in V} m_{x,y}(m_{x,y} - 1) = 0$ in definition 2.2, then either $m_{x,y} = 0$ or $m_{x,y} = 1$ for x, $y \in V$, and hence Γ is a simple graph, and moreover Γ is a strongly regular graph with parameters

$$(v, k, \lambda, \mu) = (\frac{R(R - n + 2m - \mu - 1)}{\mu} + R - 1, R, n - 2m + \mu, \mu).$$

Note that $R = \sum_{y \in V} m_{x,y}$, and $\gamma = \sum_{y \in V} m_{x,y}(m_{x,y} - 1)$ for x, $y \in V$ were defined explicitly in the definition, but there is no explicit definition on m, n and μ. Such a definition is very unnatural. Combinatorial interpretations of m, n, μ are interesting for us. The parameter μ in $SRMG(m, n, \mu, \gamma, R)$ is identical with that of μ in $SRG(v, k, \lambda, \mu)$ in case $\gamma = 0$.

The notion of strongly regular graphs can be stated in terms of the matrix.

Lemma 2.3 Let A be the adjacency matrix of a simple graph Γ, then the following are equivalent:

(1) Γ is a strongly regular graph.

(2) $AJ = kJ$, $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$.

Similar to strongly regular graphs, we want to know the *matrix expression* of strongly regular multigraph.

Lemma 2.4 Let A be the adjacency matrix of a multigraph Γ of order v, then the following are equivalent:

(1) Γ is a strongly regular graph.

(2) $AJ = RJ$, $A^2 = (n-2m)A + m(n-m)I + \mu J$, with real numbers R, m, n, μ, and $n > 0$, and $v = \frac{(R+m)(R+m-n)}{\mu}$.

Some subsets of vertices including *claws*, *maximal cliques* play essential roles in the study of their structures. We will explain in Section 4 that m is the constant number of maximal cliques containing a fix vertex in the study of maximal claws.

Definition 2.5

(1) A *clique* is a set of mutually adjacent vertices. A *maximal clique* is a clique not properly contained in any other clique.

(2) A *claw* (x, A) consists of a vertex x and an anticlique A such that x is adjacent to every vertex of A. The *order* of the claw (x, A) is defined by $\sum_{y \in A} m_{x,y}$.

It is well knows that the *block graph* of a *quasi-symmetric* 2-*design* is strongly regular. This leads to the question that whether some strongly regular multigraphs associated with some designs of various types? A class of incidence structure lies between 1-designs (regular) and 2-designs is defined. We will show in Section 4 that the *collinearity graphs* of this class of incidence structures are indeed strongly regular multigraphs.

For an incidence structures, let $m_{x,y}$ = number of blocks containing points x and y, and define $m_{x,x} = 0$.

Definition 2.6 A $1\frac{1}{2}$-design with parameter (r, k, t, c) is an incidence structure $I = (\mathcal{P}, \mathcal{B})$ such that

(1) each point x lies on r blocks;

(2) each block l contains k points;

(3) for a point x and a block l

a. $t = \sum_{y \in l} m_{x,y} \geqslant 1$ is a constant if $x \notin l$;

b. $c = \sum_{y \in l - \{x\}} (m_{x,y} - 1)$ is a constant if $x \in l$.

Let A be the incidence matrix of the incidence structure under consideration, and (x, B) is a pair of point and block, note that

$$AA^T A(x, B) = \sum A(x, B')A(y, B')A(y, B)$$

is the number of the pair (y, B') such that $x \in B'$ and $y \in B \cap B'$. The following matrix expression for $1\frac{1}{2}$-designs is immediate:

Lemma 2.7 For a binary matrix A, the following are equivalent:

(1) A is the incidence matrix of a $1\frac{1}{2}$-design.

(2) $AJ = rJ$, $JA = kJ$ and $AA^T A = (r + k - 1 + c - t)A + tJ$ for some integers r, k, t, c with $t \geqslant 1$.

Lemma 2.8 Each 2-(v, k, λ) design is a $1\frac{1}{2}$-design with parameters

$$(r, k, t, c) = (\frac{\lambda(v-1)}{k-1}, k, k\lambda, (k-1)(\lambda-1)).$$

Conversely, each $1\frac{1}{2}$-design with parameters (r, k, t, c) satisfying $(t + 1 - c - k)k = t$ is a 2-(v, k, λ) design where

$$(v, \lambda) = (1 + \frac{r(k-1)}{\lambda}, t + 1 - c - k).$$

Proof:

(1) Since the incidence matrix A of a 2-(v, k, λ) design satisfies

$$AJ = rJ,\ JA = kJ, \text{ and } AA^T = (r - \lambda)I + \lambda J,$$

multiplying the third equation by A on its both sides, we then have

$AA^TA = (r-\lambda)A + \lambda kJ$; and c is computed by $AA^TA = (r+k-1+c-t)A + tJ$.

(2) Let A be the incidence matrix of a $1\frac{1}{2}$-design with $(t+1-c-k)k = t$, and let

$$X = AA^T - (r+k+c-t-1)I + (k+c-t-1)J.$$

Then we have $X^2 = 0$, and hence $X = 0$ because X is a symmetric matrix. Hence

$$AA^T = (r+k+c-t-1)I - (k+c-t-1)J = (r-\lambda)I + \lambda J,$$

with $\lambda = t+1-c-k$. Q.E.D.

Lemma 2.9 ([9], Theorem 3.2) The collinearity graph of a $1\frac{1}{2}$-design with parameters (r,k,t,c) is a strongly regular multigraph with parameters (m,n,μ,γ,R) with

$$(m,n,\mu,\gamma,R) = (r, k+r+c-1-t, rt, rc, r(k-1)).$$

The above lemma, shows that the collinearity graph of a $1\frac{1}{2}$-design is a strongly regular multigraph. Following this trend, we are interested in those strongly regular multigraphs which are the collinearity graph of $1\frac{1}{2}$-designs or of even unique $1\frac{1}{2}$-designs? The following Theorem of Neumaier provides sufficient numerical constrains to guarantee the uniqueness of such $1\frac{1}{2}$-designs. Its proof will be given in Section 4.

Theorem 2.10 ([9], Theorem 4.4) If Γ is a strongly regular multigraph with parameters (m,n,μ,γ,R) with $m \geqslant 2$, integral $\mu \equiv 0 \bmod m$, $\mu > 0$, and

$$n > \max\{m-1+\frac{(\mu+m)\gamma}{m^2}, 2(m-1)(\mu+1-m)+2\gamma,$$
$$\frac{m(m-1)}{2}(\mu+1)+m\frac{\gamma}{2}+m-1\}$$

then Γ is the collinearity multigraph of a unique $1\frac{1}{2}$-design, with parameters

$$(r,k,t,c) = (m, \frac{R}{m}+1, \frac{\mu}{m}, \frac{\gamma}{m}).$$

There is no example of strongly regular multigraphs meeting those numerical constraints found in the papers of Bose, Neumaier and Metsch.

In the final section, we use the symmetric association scheme to define a distance regular graph, and defined a class of strongly regular multigraphs by giving the multiedge on the induce subgraph of this distance regular graph.

Definition 2.11 An *association scheme* with d classes is a finite set X together with $d+1$ relations R_i on X such that

(1) $\{R_0, R_1, ..., R_d\}$ is a partition of $X \times X$;

(2) $R_0 = \{(x, x)|x \in X\}$;

(3) for each $i \in \{0, 1, ..., d\}$ there exists a $j \in \{0, 1, ..., d\}$ such that $(x, y) \in R_i$ implies $(y, x) \in R_j$;

(4) for any $(x, y) \in R_k$ the number p_{ij}^k of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ depends only on i, j and k;

(5) $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in \{0, 1, ..., d\}$.

Moreover, if (3) and (5) replaces by

(3') if $(x, y) \in R_i$, then also $(y, x) \in R_i$, for all $x, y \in X$ and $i \in \{0, 1, ..., d\}$.

Then it is called the *symmetric association scheme.*

Definition 2.12 A *distance regular graph* is a simple graph with the intersection numbers $\{b_0, b_1, ..., b_{d-1}; c_1, c_2, ..., c_d\}$ satisfying the follows:

For all $(x, y) \in V$, if $\partial(x, y) = i$ then

(1) $c_i := |\Gamma_1(x) \cap \Gamma_{i-1}(y)|$,

(2) $b_i := |\Gamma_1(x) \cap \Gamma_{i+1}(y)|$,

(3) $a_i := |\Gamma_1(x) \cap \Gamma_i(y)|$, and

If $\partial(x, y) = 0$ then $k_i := |\Gamma_i(x) \cap \Gamma_i(y)|$.

3 Spectral of Strongly Regular Multigraphs

The matrix expressions in terms of their adjacency matrix of strong regular graph and strongly regular multigraph are quite similar. We are interested to know the spectrum of strongly regular multigraphs?

The eigenvalues of strongly regular graphs can be easily calculated in terms of the matrix equation of its adjacency matrix. We also know that the strongly regular graph are those connected regular graph with exactly three distinct eigenvalues, and the spectral of Γ is as follows:

$$spec(\Gamma) = (k^1, (\frac{1}{2}((\lambda - \mu) + \sqrt{\Delta}))^{m_1}, (\frac{1}{2}((\lambda - \mu) - \sqrt{\Delta}))^{m_2}),$$

where $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$, and

$$m_1 = (v - 1) + \frac{2k - (v - 1)(\lambda - \mu)}{\sqrt{\Delta}}$$

$$m_2 = (v - 1) + \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{\Delta}}$$

The following Lemma will prove that the converse is also true.

Lemma 3.1 A connected k-regular graph Γ is a strongly regular graph with parameters (v, k, λ, μ) if and only if it has exactly three distinct eigenvalues $k > \theta_1 > \theta_2$.

Moreover, $(\theta_1, \theta_2) = (\frac{1}{2}((\lambda-\mu)+\sqrt{\Delta}), \frac{1}{2}((\lambda-\mu)-\sqrt{\Delta}))$ where $\Delta = (\lambda-\mu)^2+4(k-\mu)$.

Lemma 3.1 can be extended to strongly regular multigraphs with a mimic proof. Before shown the Lemma 3.3, we need some technique. First, we need to make sure if Γ is a connected R-regular multigraph, then R is also an eigenvalue. And next we need make sure the multiplicity of the corresponding eigenvalue R is 1.

Proposition 3.2 Let Γ be a multigraph,

(1) Γ is R-regular multigraph if and only if the largest absolute eigenvalue of Γ is R.

(2) The multiplicity of R as an eigenvalue is 1 if Γ is connected.

Proof: Let A be the adjacency matrix of Γ. Take $x = (x_1, x_2, ..., x_v)^T$ be an eigenvector for eigenvalue λ, and let x_i be a coordinate of largest absolute value among coordinates of x. For the i-th coordinate of Ax, we have

$$|\lambda||x_i| = |(Ax)_i| = |\sum_{j=1}^{v} A_{ij}x_j| \leqslant |\sum_{j=1}^{v} A_{ij}x_i| \leqslant |\sum_{i\sim j} A_{ij}||x_i| = R|x_i|.$$

Hence $\lambda \leqslant R$. Equality requires $x_j = x_i$ for all $x_j \in N(x_i)$. We can iterate this argument to reach all coordinates for vertices in Γ. Thus, the multiplicity of R is 1. Q.E.D.

Lemma 3.3 A connected R-regular multigraph Γ is a strongly regular multigraph with parameters (m, n, μ, γ, R) if and only if it has exactly three distinct eigenvalues $R > \theta_1 > \theta_2$. Moreover, $spec(\Gamma) = (R^1, (n-m)^{t_1}, (-m)^{t_2})$ where

$$(t_1, t_2) = (\frac{m(v-1)-R}{n}, \frac{(n-m)(v-1)-R}{n}).$$

Proof:

First, we assume that Γ is a strongly regular multigraph with an adjacency matrix A, then $A^2 = (n-2m)A + m(n-m)I + \mu J$. Multiplying by A on both sides, we have

$A^3 = (n-2m+R)A^2 - (R(n-2m) - m(n-m))A - mR(n-m)I$, that is,

$$A^3 - (n-2m+R)A^2 + (R(n-2m) - m(n-m))A + mR(n-m)I = 0.$$

Hence the minimal polynomial of A is given by

$$f(x) = (x-R)(x-(n-m))(x+m).$$

So the strongly regular graph has three distinct eigenvalues R, $n-m$, and $-m$, let 1, t_1, and t_2 be their multiplicities respectively. Since the trace of A equal to the sum of all eigenvalues, and the number of eigenvalues are equal to the number of vertices. So we have

$$(t_1, t_2) = (\frac{m(v-1)-R}{n}, \frac{(n-m)(v-1)-R}{n}).$$

On the other hand, let $R > \theta_1 > \theta_2$ be the three distinct eigenvalues, since the multiplicity of R is 1 because A is the adjacency matrix of a connected regular multigraph. Define

$$M := \frac{1}{(R-\theta_1)(R-\theta_1)}(A-\theta_1 I)(A-\theta_2 I).$$

Since A and A^2 so symmetric. Take x is an eigenvector of A with the corresponding eigenvalues θ_1 (or θ_2 respectively), then $Mx = 0$, i.e., x is an eigenvector of M with eigenvalues 0. Thus all eigenvectors of corresponding eigenvalue is 0 are in the kernel of M. Thus, the rank of M is 1, equal to the multiplicity of R. Then we have

$$M = \begin{pmatrix} a_1 & a_2 & \cdots & a_v \\ a_1 & a_2 & \cdots & a_v \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_v \end{pmatrix} \text{ or } \begin{pmatrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_v & a_v & \cdots & a_v \end{pmatrix}$$

Furthermore, M is symmetric, hence $a_1 = a_2 = \cdots = a_v = a$ is a constant. Let $\vec{1}$ be the all one vector, then we have $M\vec{1} = \vec{1}$. Hence $M = \frac{1}{v}J$, that is A^2 is a linear combination of A, J and I. Thus A is the adjacency matrix of some strongly regular multigraphs. Q.E.D.

Remark: From above proof, we have $n - m < R$ because the multiplicity of R is 1.

Form Lemma 3.1 and Lemma 3.3, we know that a connected k-regular graph (respectively, multigraph) with exactly three distinct eigenvalues k, θ_1 and θ_2 is a strongly regular graph (respectively, a strongly regular multigraph). Moreover,

$$\frac{1}{v}J = \frac{1}{(k-\theta_1)(k-\theta_2)}(A-\theta_1 I)(A-\theta_2 I),$$

that is,

$$A^2 = (\theta_1+\theta_2)A - \theta_1\theta_2 I + \frac{1}{v}(k-\theta_1)(k-\theta_2)J.$$

Consider that $(\theta_1+\theta_2)A-\theta_1\theta_2 I+\frac{1}{v}(k-\theta_1)(k-\theta_2)J = (n-2m)A+m(n-m)I+\mu J$ for some m, n, μ, γ with $n > 0$ are real numbers then we can compute it directly and then have

$$(m,n,\mu,\gamma,R) = (-\theta_2, \theta_1-\theta_2, \frac{(k-\theta_1)(k-\theta_2)}{v}, \frac{(k-\theta_1)(k-\theta_2)}{v} - k - \theta_1\theta_2, k).$$

If the graph is the simple graph, then we have

$$(v,k,\lambda,\mu) = (\frac{(k-\theta_1)(k-\theta_2)}{k+\theta_1\theta_2}, k, \theta_1\theta_2+\theta_1+\theta_2+k, k+\theta_1\theta_2).$$

Since $\gamma = \sum_{y\sim x} m_{x,y}(m_{x,y}-1) = \frac{(k-\theta_1)(k-\theta_2)}{v} - k - \theta_1\theta_2 \geqslant 0$, we have $(k-\theta_1)(k-\theta_2) \geqslant v(k+\theta_1\theta_2)$. The case equality or otherwise correspond to strongly regular graphs or strongly regular multigraphs respectively.

Note: As $\gamma = 0$, Γ is a strongly regular graph with parameters

$$(v,k,\lambda,\mu) = (\frac{R(R-n+2m-\mu-1)}{\mu} + R - 1, R, n-2m+\mu, \mu).$$

Then we have

$$(m,n,\mu,\gamma,R)=(\frac{1}{2}(\sqrt{\Delta}-(\lambda-\mu)),\sqrt{\Delta},\mu,0,k),\ \text{where}\ \Delta=(\lambda-\mu)^2+4(k-\mu).$$

Then

$$spec(\Gamma)=(R^1,(n-m)^{t_1},(-m)^{t_2}),$$

where

$$(t_1,t_2)=(\frac{m(v-1)-R}{n},\frac{(n-m)(v-1)-R}{n}),$$

can be reduce to the strongly regular graph with

$$spec(\Gamma)=(k^1,(\frac{1}{2}((\lambda-\mu)+\sqrt{\Delta}))^{m_1},(\frac{1}{2}((\lambda-\mu)-\sqrt{\Delta}))^{m_2})$$

where $\Delta=(\lambda-\mu)^2+4(k-\mu)$ and

$$(m_1,m_2)=((v-1)+\frac{2k-(v-1)(\lambda-\mu)}{\sqrt{\Delta}},(v-1)+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{\Delta}}).$$

Then we have the following:

Theorem 3.4 If A is an adjacency matrix of order v of a k-regular (either simple or multiple) graph with three distinct eigenvalues $k>\theta_1>\theta_2$, then $(k-\theta_1)(k-\theta_2)\geqslant v(k+\theta_1\theta_2)$. Moreover,

(1) if $(k-\theta_1)(k-\theta_2)=v(k+\theta_1\theta_2)$, then the graph is a strongly regular graph $SRG(v,k,\lambda,\mu)$ with

$$(v,k,\lambda,\mu)=(\frac{(k-\theta_1)(k-\theta_2)}{k+\theta_1\theta_2},k,\theta_1\theta_2+\theta_1+\theta_2+k,k+\theta_1\theta_2);$$

(2) if $(k-\theta_1)(k-\theta_2)>v(k+\theta_1\theta_2)$, then the graph is nontrivial strongly regular multigraph $SRMG(m,n,\mu,\gamma,R)$ with

$$(m,n,\mu,\gamma,R)=(-\theta_2,\theta_1-\theta_2,\frac{(k-\theta_1)(k-\theta_2)}{v},\frac{(k-\theta_1)(k-\theta_2)}{v}-k-\theta_1\theta_2,k).$$

Proof: From the definition $\gamma = \sum_{y \sim x} m_{x,y}(m_{x,y} - 1)$, we have

$$(k - \theta_1)(k - \theta_2) \geqslant v(k + \theta_1\theta_2).$$

(1) If $\gamma = 0$, then A must be the adjacency matrix of a strongly regular graph.

(2) If $\gamma > 0$, then A must be the adjacency matrix of a strongly regular multigraph.

Q.E.D.

A comparison between strongly regular graphs and strongly regular multigraphs are included in the following table:

	$SRG(v, k, \lambda, \mu)$	$SRMG(m, n, \mu, \gamma, R)$
adjacency matrix	a symmetric (0,1)-matrix	a symmetric matrix with nonnegative entries
adjacency matrix expression	$AJ = kJ$, $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$	$AJ = RJ$, $A^2 = (n - 2m)A + m(n - m)I + \mu J$
relative design	quasi-symmetric 2-design	$1\frac{1}{2}$-design
three distinct eigenvalues	k, $\frac{1}{2}((\lambda - \mu) \pm \sqrt{\Delta})$, where $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$	R, $n - m$, $-m$
given $k > \theta_1 > \theta_2$	$v = \frac{(k-\theta_1)(k-\theta_2)}{k+\theta_1\theta_2}$, $k = k$, $\lambda = \theta_1\theta_2 + \theta_1 + \theta_2 + k$, $\mu = k + \theta_1\theta_2$	$m = -\theta_2$, $n = \theta_1 - \theta_2$ $\mu = \frac{(k-\theta_1)(k-\theta_2)}{v}$ $\gamma = \frac{(k-\theta_1)(k-\theta_2)}{v} - k - \theta_1\theta_2$, $R = k$
	$(k - \theta_1)(k - \theta_2) = v(k + \theta_1\theta_2)$	$(k - \theta_1)(k - \theta_2) > v(k + \theta_1\theta_2)$

Table 1

4 A Survey of Papers of Bose, Neumaier and Metsch

The subjects of strongly regular multigraph were thoroughly studied in the papers of R. C. Bose ([2], 1976), Neumaier ([9], 1982) and Metsch ([7], 1995). A study of these three papers will be given in this section in an unified way.

These three papers share some common ground with various terminologies, there facts are listed in Table 2 for reference. The results, and the ways of their proofs for each papers were given in subsections 4-2∼4-4 respectively.

The content of subsections 4-2∼4-4 are in the following order: First, the main theorems concerning the quasi-residual designs are gives; following by their proofs in sketch the strategies for the proofs of these main theorem and provided in figures 1∼3.

The relation of Bose, Neumaier, and Metsch's theorem:

Bose	Neumaier	Metsch
[2], Lemma 2.1 (lower bound) [2], Theorem 2.8 (unique)	[9], Lemma 2.3 (upper bound)	
[2], Theorem 2.5	[9], Lemma 4.1(b)	
[2], Theorem 2.6	[9], Lemma 4.1(c)	
[2], Theorem 2.7	[9], Lemma 4.1(c)	
[2], Theorem 3.3(1)		[7], Lemma 3.22
[2], Theorem 3.3(2)	[9], Lemma 4.2	[7], Lemma 2.10
[2], Theorem 3.3(3)		[7], Lemma 3.10
[2], Lemma 3.3		[7], Lemma 3.11
[2], Theorem 4.2	[9], Theorem 3.5	
[2], Theorem 4.1	[9], Theorem 1.2(2)	
[2], Theorem 4.3	[9], Theorem 1.3	

Table 2

4.1 The 1976 paper of Bose

A multigraph G is said to be *regular* of *degree* n and *loop degree* d if $n(x) := \sum_{y\sim x} m_{x,y} = n$ and $d(x) := \sum_{y\neq x} \frac{(m_{x,y})(m_{x,y}-1)}{2} = d$ are constant for each vertex x in G.

A regular multigraph G is called *edge regular* if for any set of adjacent vertices x and y, the quantity $p(x,y) := \sum_{z\sim x,y} m_{x,z}m_{y,z}$ depends only on the multiplicity $m_{x,y}$ of x and y. The concept of "*type*" was defined by Bose over edge regularity of multigraphs.

An edge regular multigraph will be said to be of the type $G_k\{r,d;\alpha_0,\alpha_1,\ldots\alpha_r\}$ if it satisfies the following properties:

(1) $m_{x,y} \leqslant r$ for any edge xy.

(2) The *degree* $n(x)$ of any vertex is given by $n(x) = r(k-1)$.

(3) The *loop degree* $d(x) = d$, for any vertex x.

(4) For any edge xy for which $m_{x,y} = m \geqslant 1$, $p(x,y) = m(k-2)$ is a divisor of α_m.

(5) If the vertices, x and y are nondajacent, i.e., $m_{x,y} = 0$, $p(x,y) \leqslant \alpha_0$.

Furthermore, he proposed the definition of strongly regular multigraphs over the edge regular graphs. Obviously, the difference between edge regular and strongly regular multigraph is that the edge regular posed condition over pairs of adjacent vertices depends on the multiplicity $m_{x,y}$, but strongly regular multigraphs posed conditions over pair of distinct vertices depends on the multiplicity $m_{x,y}$.

Then Bose gave a definition of a *claw*. A claw (x,S) of the multigraphs G_k is defined as a set of vertices $S = \{y_1, y_2, \ldots, y_s\}$ nonadjacent to each other and each adjacent to a vertex x, i.e., $m_{x,y_i} = m_i \geqslant 1$ and $m_{y_i,y_j} = 0$ if $i \neq j = 1, 2, \ldots, s$. A claw (x, S) is said to be of type $(a_1, a_2, \ldots, a_r)$ if a_i be the number of edges xy, $y \in S$ for which $m_{x,y} = i$. With the claw we can associate four parameters (s, μ, δ, π) define as follows:

(1) $s = \sum_{i=1}^{r} a_i$, the *order of the claw.*

(2) $\mu = \sum_{i=1}^{r} i a_i = \sum_{i=1}^{s} m_i$, the *multiplicity of the claw.*

(3) $\delta = \frac{1}{2}\sum_{i=1}^{r} i(i-1)a_i = \frac{1}{2}\sum_{i=1}^{s} m_i(m_i - 1)$, the *loop multiplicity* of the claw.

(4) $\pi = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{s} \alpha_{m_i}$, the *coefficient of edge regularity* of the claw.

He defined *grand cliques* and then studied the structures strongly regular multigraph in term of their claws and grand cliques.

4.2 The 1982 paper of Neumaier

For any two distinct blocks $A, B \neq H$, where H is a block, denote by $\alpha_{A,B}$ the number of points in H incident with A and B, and $\beta_{A,B}$ the number of points not in H incident with A and B.

Theorem 4.2.1 ([9], Theorem 1.2) A quasi-residual 2-(v, k, λ)-design $\pi = (\mathcal{P}, \mathcal{B})$ is embeddable if and only if the following three conditions are satisfied:

(1) Any two distinct blocks A and B intersect in $\beta_{A,B} \leqslant \lambda$ points.

(2) The multigraph G defined over the blocks, with $\alpha_{A,B} = \lambda - \beta_{A,B}$ edges between A and B, is a strongly regular multigraph with parameters

$$(m, n, \mu, \gamma, R) = (\lambda, k, \lambda^2(\lambda - 1), \lambda(\lambda - 1)(\lambda - 2), \lambda(k + \lambda - 2)).$$

(3) G is isomorphic to the block multigraph of a 2-$(n+\lambda, \lambda, \lambda-1)$-design $\pi' = (\mathcal{P}', \mathcal{B}')$.

Theorem 4.2.2 ([9], Theorem 1.3) A quasi-residual 2-(v, k, λ)-design $\pi = (\mathcal{P}, \mathcal{B})$ is embeddable if either $\lambda = 3$, and $k > 76$, or $\lambda \neq 3$, and $k > \frac{1}{2}(\lambda^2 - 1)(\lambda^3 - \lambda^2 - \lambda + 2)$.

The main theorem shows that under what numerical constraints, a strongly regular multigraph will be the point graph of a unique $1\frac{1}{2}$-design (Theorem 4.2.17). To prove it, Neumaier first showed that this strongly regular multigraph is the *point graph* of a *weak* $1\frac{1}{2}$-*design*, and then showed further that it is a $1\frac{1}{2}$-*design*. The uniqueness of this $1\frac{1}{2}$-design is guaranteed by showing that each block is a *grand clique* of the strongly regular multigraph under consideration.

Step 1: Show first that there is no s-claw (x, S) whenever $s > m$ in terms of the quantity $N = \sum_{x \neq y}(\alpha_x - m_{y,x})(\alpha_x - m_{y,x} - 1)$ where $\alpha_x := \sum_{y \in S} m_{x,y}$, and the contradictory argument related to upper and lower bounds (Lemma 4.2.14).

Step 2: Show further that each point is in exactly m grand cliques, and each edge $\overline{ab}$ of multiplicity $m_{a,b}$ is in exactly $m_{a,b}$ cliques by a constructive argument (Lemma 4.2.15).

Step 3: Show that a $SRMG$ is a point graph of a weak $1\frac{1}{2}$-design if and only if the two condition satisfied.

(1) there is a collection $\sum$ of cliques such that every point is in exactly m cliques of $\sum$, and

(2) every edge $\overline{ab}$ of multiplicity $m_{a,b}$ is in exactly $m_{a,b}$ cliques of $\sum$ (Theorem 4.2.12) in terms of incidence matrices of designs and the adjacency matrices of strongly regular multigraphs.

Step 4: Show the constant size of blocks (Theorem 4.2.13) under either of the following conditions

(1) two distinct points are in at most one blocks,

(2) $t = \frac{\lambda(\lambda v+n)}{r}$ is an integer with $\lambda(n+1-r) < (1-\lambda)(t+1)$ in terms of the quantity

$s(x, B) = |\{(a, A)|(a, A) \in \mathcal{P} \times \mathcal{B}, x, a \in A, a \in B\}|$.

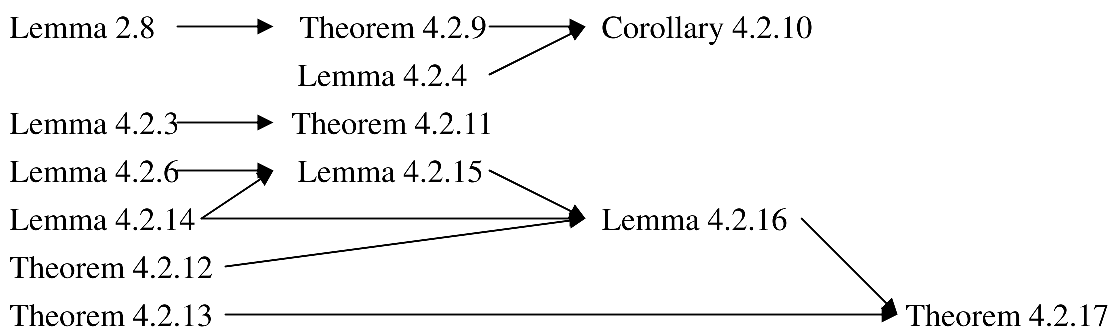

Figure 1: [9], Neumaier

Lemma 4.2.3 ([9], Lemma 2.1) Let A be an integral symmetric matrix with zero diagonal satisfying $AJ = RJ$, $A^2 = (n-2m)A + m(n-m)I + \mu J$. If $n > \max\{2m-4, 2m-1+\mu+\gamma\}$ where $\gamma = m(n-m) + \mu - R$, then A is the adjacency matrix of a strongly regular multigraph with parameters (m, n, μ, γ, R).

Lemma 4.2.4 ([9], Lemma 2.2) Let Γ be a strongly regular multigraph graph with parameters (m, n, μ, γ, R).

(1) $m - n \leqslant m_{a,b} \leqslant m$.

(2) $m \geqslant 1$, with equality if and only if Γ is the disjoint union of complete graphs.

(3) If there are nonadjacent points then $n \geqslant m$.

(4) $\mu \geqslant (R+m)(m-n)$, with equality if and only if $m_{a,b} = m - n$ for all $a \neq b$.

(5) $\mu\gamma \leqslant (n - 2m + \mu)(m(n-m) + \mu)$, with equality if Γ contains no triangles.

(6) $\mu \geqslant 2m - n$.

(7) If $n \leqslant 2m+4$, then $\gamma < 2m(n-m)+n-2m-1+\mu$.

Definition 4.2.5 A maximal clique C with $|C| > \frac{n}{2}+\mu+1-m$ is called a *grand clique*.

Lemma 4.2.6 ([9], Lemma 2.3) An edge of multiplicity one is in at most one grand clique.

Definition 4.2.7 An incidence structure with an incidence matrix A is a *weak* 2-*design* if $AJ = rJ$, $AA^T = nI+\lambda J$ and a *weak* $1\frac{1}{2}$-*design* if $AJ = rJ$ and $AA^TA = nA+\lambda JA$.

Theorem 4.2.8 ([9], Theorem 3.2) The *point multigraph* of a weak $1\frac{1}{2}$-design with parameters (v,n,r,λ) is a strongly regular multigraph with parameters

$$(m,n,\mu,\gamma,R) = (r,n,\lambda(\lambda v+n),\lambda(\lambda-1)v-r(r-1)+(r+\lambda-1)n,\lambda v-n+r).$$

In particular, the point multigraph of a $1\frac{1}{2}$-design with parameters (r,k,t,c) is a strongly regular multigraph with parameters

$$(m,n,\mu,\gamma,R) = (r,r+k+c-1-t,rt,rc,r(k-1))$$

$$\text{and } (r,k,t,c) = (m,\frac{R}{m}+1,\frac{\mu}{m},\frac{\gamma}{m}).$$

Theorem 4.2.9 ([9], Theorem 3.3) The block multigraph of a 2-(v,k,λ)-design of order n is a strongly regular multigraph with parameters

$$(m,n,\mu,\gamma,R) = (k,n,k^2\lambda,k(k-1)(\lambda-1),k(n+\lambda-1)).$$

Corollary 4.2.10 ([9], Corollary 3.4) Two distinct blocks A and B of a 2-(v,k,λ)-design intersect in at least $k-r+\lambda$ points.

Theorem 4.2.11 ([9], Theorem 3.5) Let $\pi = (\mathcal{P}, \mathcal{B})$ be a quasi-residual 2-(v, k, λ)-design with $k > 2\lambda^3 - 4\lambda^2 + 4\lambda - 1$. Then

(1) two distinct blocks intersect in at most λ points, and

(2) the multigraph Γ on the blocks, with $\alpha_{A,B} = \lambda - \beta_{A,B}$ edges between A and B, is a strongly regular multigraph with parameters

$$(m, n, \mu, \gamma, R) = (\lambda, k, \lambda^2(\lambda - 1), \lambda(\lambda - 1)(\lambda - 2), \lambda(k + \lambda - 2)).$$

Theorem 4.2.12 ([9], Theorem 3.6) A strongly regular multigraph Γ with parameters (m, n, μ, γ, R) is the point multigraph of a weak $1\frac{1}{2}$-design if and only if there is a collection $\sum$ of cliques such that every point is in exactly m cliques of $\sum$, and every edge $\overline{ab}$ of multiplicity $m_{a,b}$ is in exactly $m_{a,b}$ cliques of $\sum$. Moreover, the blocks are the cliques of $\sum$, and the weak $1\frac{1}{2}$-design has parameters

$$(v, n, r, \lambda) = (\frac{(R + m)(R + m - n)}{\mu}, n, m, \frac{\mu}{R + m}).$$

Theorem 4.2.13 ([9], Theorem 3.7) Let $\pi = (\mathcal{P}, \mathcal{B})$ be a weak $1\frac{1}{2}$-design with parameters (v, n, r, λ), and $\lambda < 1$. If either

(1) two distinct points are in at most one blocks, or

(2) $t = \frac{\lambda(\lambda v + n)}{r}$ is an integer with $\lambda(n + 1 - r) < (1 - \lambda)(t + 1)$,

then $\pi = (\mathcal{P}, \mathcal{B})$ is a $1\frac{1}{2}$-design, with parameters

$$(r, k, t, c) = (r, \frac{\lambda v + n}{r}, \frac{\lambda(\lambda v + n)}{r}, n + 1 + t - r - k).$$

Lemma 4.2.14 ([9], Lemma 4.1) Let Γ be a strongly regular multigraph with parameters (m, n, μ, γ, R) with $\mu \geqslant 1$, and integral $m \geqslant 2$. The following hold:

(1) If $2n > m(m-1)(\mu+1) + m\gamma + 2m - 2$, then $s \leqslant m$ for every s-claw.

(2) If $2n > (m-3)(\mu - m) + 2\gamma + 2m - 2$ then every s-claw $(1 \leqslant s \leqslant m-2)$ can be extended to a $(s+1)$-claw.

(3) If there are no s-claw with $s > m$, then every $(m-1)$-claw is in at least $n - 1 - (m-2)(\mu + 1 - m) - \gamma$ many of m-claws.

(4) If (a, S) is a maximal m-claw, then there are at least $m(n-2) - (m-2)\mu - 2\gamma$ many of m-claws (a, S') such that $|S \cap S'| = m - 1$.

Lemma 4.2.15 ([9], Lemma 4.2) Let Γ be a strongly regular multigraph with parameters (m, n, μ, γ, R) with $\mu \geqslant 1$, and integral $m \geqslant 2$. If there are no s-claws with $s > m$ and if

$$n > \max\{\frac{1}{2}(m-3)(\mu - m) + \gamma + m - 1, 2(m-1)(\mu + 1 - m) + 2\gamma\}$$

then each edge $\overline{ab}$ of multiplicity $m_{a,b}$ is exactly $m_{a,b}$ cliques, and each point is in exactly m grand cliques.

Lemma 4.2.16 ([9], Lemma 4.3) If Γ is a strongly regular multigraph with parameters (m, n, μ, γ, R) with $\mu \geqslant 1$, and integral $m \geqslant 2$, and

$$n > \max\{2(m-1)(\mu + 1 - m) + 2\gamma, \frac{1}{2}m(m-1)(\mu+1) + \frac{1}{2}m\gamma + m - 1\}$$

then Γ is the point multigraph of weak $1\frac{1}{2}$-design with parameters (v, n, r, λ) given by

$$(v, n, r, \lambda) = (\frac{(R+m)(R+m-n)}{\mu}, n, m, \frac{\mu}{R+m}).$$

Theorem 4.2.17 ([9], Theorem 4.4) If Γ is a strongly regular multigraph with parameters (m, n, μ, γ, R) with integral $m \geqslant 2$, integral $\mu \equiv 0 \bmod m$, $\mu > 0$, and

$$n > \max\{m - 1 + \frac{(\mu+m)\gamma}{m^2}, 2(m-1)(\mu+1-m) + 2\gamma, \frac{m(m-1)}{2}(\mu+1) + m\frac{\gamma}{2} + m - 1\}$$

then Γ is the point multigraph of unique $1\frac{1}{2}$-design, with parameters

$$(r, k, t, c) = (m, \frac{R}{m} + 1, \frac{\mu}{m}, \frac{\gamma}{m}).$$

Corollary 4.2.18 ([9], Corollary 4.5) Every strongly regular multigraph with parameters

$$(m, n, \mu, \gamma, R) = (k, n, k^2\lambda, k(k-1)(\lambda-1), k(n+\lambda-1))$$

such that n, k, λ are positive integers, $k \neq 1$ and

$$n > \max\{k(k-1)\lambda^2 - (k-1)^2\lambda, 2(k-1)(k^2\lambda + k\lambda - 2\lambda + 1), \frac{1}{2}(k^2-1)(k^2\lambda - k + 2)\}$$

is isomorphic to the block multigraph of a 2-(v, k, λ)-design with $v = k + \lambda^{-1}n(k-1)$.

Theorem 4.2.19 ([9], Theorem 4.6) If Γ is a strongly regular multigraph with parameters (m, n, μ, γ, R) with $\mu \geqslant 1$, integral m, $2 \leqslant m \leqslant n$, and

$$n > \max\{2(m-1)(\mu+1-m), \frac{m(m-1)}{2}(\mu+1) + m - 1\}$$

then Γ is the point graph of a unique partial geometry with parameters

$$(r, k, t) = (m, \frac{R}{m} + 1, \frac{\mu}{m}).$$

4.3 The section 2 of 1995 paper of Metsch

Theorem 4.3.1 ([7], Theorem 2.1) A strongly regular multigraph with parameters (m, n, μ, γ, R) with $\mu, m \geqslant 2$. Suppose

(1) $n > 2(m + a - 2)(\mu - 1) + 3\gamma$,

(2) $2(a+1-m)n > a(a-1)\mu + a\gamma + 4m(a+1) - 2m^2 - (a+1)(a+2)$

for some $a \geqslant m$, then the family $M = \{C|C$ is a maximal clique with $|C| \geqslant n-2m-(a-2)\mu - \frac{3}{2}\gamma + a + 1\}$ satisfies

(a) For each vertex x and if a_x is maximum order of a claw (x, A), then $m \leqslant a_x \leqslant a$ and there exists exactly a_x cliques in M which contains x.

(b) If x, y are adjacent, then x and y lies in at least $m_{x,y}$ and at most $2m_{x,y}-1$ cliques of M.

(c) If x, y are adjacent and x lies in exactly m cliques of M, then there exists exactly $m_{x,y}$ cliques in M contains x and y.

Definition 4.3.2 A maximal clique C with $|C| \geqslant n - 2m - (a-2)\mu - \frac{3}{2}\gamma + a + 1$ is called a *normal clique.*

To derive each pair of adjacent vertices x and y are contained in exactly $m_{x,y}$ normal cliques (Lemma 4.3.9) whenever the maximal of order a_x of a claw (x, A) is m. It will then be used to ensure condition (c) for Theorem 4.4.1.

Toward this goal, we first study lower bounds and upper bounds for the number of normal cliques containing a vertex (indeed exactly bound, Lemma 4.3.1 (1)), or containing a pair of adjacent vertices respectively (Lemma 4.3.4) in terms of some functions for counting purpose.

a. each vertex is adjacent to at least $R - \frac{1}{2}\gamma$ vertices;

b. each vertex x is adjacent to at most $\frac{1}{2}\gamma$ vertices y with $m_{x,y} \geqslant 2$;

c. any two distinct vertices x, y have at least $w_{x,y} - \frac{3}{2}\gamma$ common neighborhoods, where $w_{x,y} = \sum_{z \in V} m_{x,z} m_{z,y}$.

For a maximal claw (x, A) with order a_x, and a point $y \in A$ adjacent to x:

We first show that there are at most $2m_{x,y} - 1$ cliques containing adjacent vertices x and y (Lemma 4.3.8 (2)) by applying the principle of inclusion and exclusion to derive upper bound and lower bound of $\sum_{i=1}^{2m_{x,y}} |C_i - \{x, y\}|$ respectively in terms of the quantity $\sum_{i=1}^{2m_{x,y}} |C_i|$ for $2m_{x,y}$ maximal cliques containing x and y, together with a hypothesis condition in Theorem 4.3.1.

If $m_{x,y} = 1$, Metsch showed that

$$C_y := \{x, y\} \cup \{z | z \notin A \in A, x \sim y, y \sim z, \forall w \in A - \{y\}, w \nsim z\},$$

lies in exactly one normal clique (Lemma 4.3.7 (1)) by showing the size of C_y meeting the require condition for normal cliques in terms of the maximality of (x, A) and the bounds (Lemma 4.3.4) for neighbors. We then show that C_y meets A nontrivially (Lemma 4.3.6). Indeed, it is true for any normal clique containing x by considering upper bounds of $|C|$.

On the other hand, for $y \in A$ with $m_{x,y} > 1$, then A can be replaced by another maximal claw (x, A') with $A' = A - \{y\} \cup \{z_1, z_2, \ldots z_{m_{x,y}}\}$ with order a_x and $m_{x,z_i} = 1$ for each i (Lemma 4.3.7 (2)) by showing the existence of exactly pairwise nonadjacent $m_{x,y}$ common neighbors of x and y outside A and not adjacent to any vertices in A except y in terms of the bound for neighbors (Lemma 4.3.4). Continuing this process, any maximal claw (x, A) can be replaced by another maximal claw (x, A') with $m_{x,z} = 1$ for each $z \in A'$.

There is a normal clique C_y meeting A' nontrivially for each $y \in A'$, it follows that there are a_x normal cliques containing x. We conclude that there exist exactly

a_x cliques contains x for all $x \in V$ (Lemma 4.3.8 (1)) and then there are at least $m_{x,y}$ normal cliques containing adjacent pair x, y of points (Lemma 4.3.8 (1)).

Finally, we claim that $a_x \geqslant m$ (Lemma 4.3.5 (2)) by considering $|\{z|z \sim x, z \sim y \in A\}| + |A|$. We conclude that each pair of adjacent vertices x and y are contained in exactly $m_{x,y}$ normal cliques (Lemma 4.3.9) by combining Lemma 4.3.8 (1) and Lemma 4.3.5 (2). The hypotheses of Theorem 4.3.1 is guaranteed under the condition given in Lemma 4.3.3.

Lemma 4.3.3 ([7], Lemma 2.11) If $m \geqslant 2$, $\mu \geqslant m, \gamma + 1$ and $n > 2(1 + \frac{2}{\sqrt{3}})m\mu = \frac{2(3+2\sqrt{3})}{3}m\mu \approx 4.3m\mu$, then the hypotheses of Theorem 4.3.1 are satisfied for the unique integer a satisfying $\frac{2}{\sqrt{3}}m - 1 < a \leqslant \frac{2}{\sqrt{3}}m$.

Theorem 4.3.5 is proved by Lemmas 4.3.4 $\sim$ 4.3.9, and the relationship between Lemma 4.3.4 $\sim$ Lemma 4.3.9 are given below:

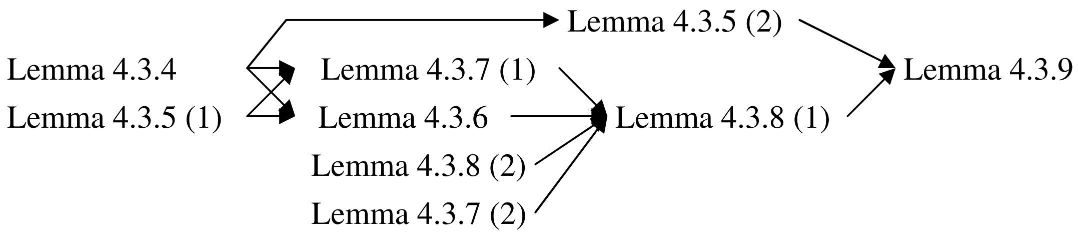

Figure 2: Section 2 in [7], Metsch

Lemma 4.3.4 ([7], Lemma 2.2)

(1) Each vertex x is adjacent to at most $\frac{1}{2}\gamma$ vertices y with $m_{x,y} \geqslant 2$;

(2) Each vertex is adjacent to at least $R - \frac{1}{2}\gamma$ vertices;

(3) Any two distinct vertices x, y have at least $w_{x,y} - \frac{3}{2}\gamma$ common neighborhoods.

Lemma 4.3.5

(1) Every claw has order at most a. ([7], Lemma 2.4)

(2) Every maximal claw has order at least m. ([7], Lemma 2.9)

Lemma 4.3.6 ([7], Lemma 2.6) If (x, A) is a maximal claw, and C is a normal clique containing x, then $A \cap C \neq \phi$.

Lemma 4.3.7 For a claw (x, A) of order a_x and a vertex $y \in A$:

(1) if $m_{x,y} = 1$, then the set,

$C := \{x, y\} \cup \{z \notin A$ adjacent y but not to any vertex of $A - \{y\}\}$

is contained in a normal clique. ([7], Lemma 2.5)

(2) if $s := m_{x,y} \geqslant 2$, then there are mutually non-adjacent vertices $y_1, y_2, \ldots y_s$ satisfying $m_{x,y_i} = 1$ and such that each y_i is adjacent to y but not to any other vertex of A. ([7], Lemma 2.7)

Lemma 4.3.8 ([7], Lemma 2.8)

(1) Each vertex x lies on exactly a_x normal cliques.

(2) Any two adjacent vertex x and y lies in at most $2m_{x,y} - 1$ cliques ([7], Lemma 2.3), and lies in at least $m_{x,y}$ normal cliques.

Lemma 4.3.9 ([7], Lemma 2.10) For a claw (x, A) with maximum order m, then x and any $y(\neq x)$ are in exactly $m_{x,y}$ normal cliques.

4.4 The section 3 of 1995 paper of Metsch

Theorem 4.4.1 ([7], Theorem 2) Suppose that Γ is a strongly regular multigraph with parameters (m, n, μ, γ, R) with geometric parameter (r, k, t, c), i.e., $(m, n, \mu, \gamma, R) = (r, n, rt, rc, r(k-1))$ with integers $r \geqslant 3$ and $t \geqslant 1$, and real numbers $k > 0$ and $c \geqslant 0$. If $k > (\frac{8}{\sqrt{3}}r + r + 5)rt$, $k > (c+1)t$, and $r(c + r - 1) \leqslant (r-1)t$, then Γ is the *point graph* of a $1\frac{1}{2}$-design with parameters (r, k, t, c).

We will show $\mu_x = r(= m)$, where $\mu_x :=$ number of cliques contains x, for each vertex x (Lemma 4.4.11) by showing that the set $V_s = \{x \in V | \mu_x > r\}$ is empty. Toward this goal, we will find an upper bound for $\sum_{x \in V} s_x$ where $s_x := \sum_{x \in C} \sum_{C \neq C'} |C \cap C'|^2$.

We then derive Lemmas 4.4.2 $\sim$ 4.4.7 in terms of elementary counting techniques and taking advantage of some inequalities.

Find an upper bound for $\sum_{i=1}^{\mu_x}(|C_i| - 1)$, and in particular its exact value whenever $\mu_x = r$ (Lemma 4.4.2), and upper bound for the size of maximal cliques (Lemma 4.4.4) and their lower bound when they contain a vertex x with $\mu_x = r$ (Lemma 4.4.3). We then find an upper bound for $\sum_{C \neq C' \in M} |C \cap C'|^2$ in terms of $\sum_{x \in V}(\mu_x - r)$ (Lemma 4.4.5). Moreover, an upper bound for $\sum_{x \in V}(s_x - 3rtk(\mu_x - r))$ (Lemma 4.4.6), lower bounds for $\mu_x s_x$ in terms of $\sum_{i=1}^{\mu_x}(|C_i| - 1)^2$ and $\max\{|C_i| | i \leqslant \mu_x\}$ (Lemma 4.4.7 (1,2)) respectively; also lower bound for s_x (Lemma 4.4.7 (3)) and for $\mu_x s_x$ (Lemma 4.4.7 (4)) for vertex x with $\mu_x = r$, and $\mu_x > r$ respectively.

We show that there exist at most $2rk$ vertices x satisfying $\mu_x > r$ (i.e., $|V_s| = v_s \leqslant 2rk$, Lemma 4.4.8) by studying upper and lower bounds of $t_x - s_0$ where $t_x := s_x - 3rtk(\mu_x - r)$ and $s_0 := (r-1)tk - (r-1)t^2$. We then show that each point x with $\mu_x > r$ lies in at least five normal cliques consisting of all points y with $\mu_y > r$ (Lemma 4.4.9) by a contradictory argument together with some inequalities.

Moreover, we show $|C_1 \cap C_2| \leqslant \mu$ for any distinct maximal cliques C_1, C_2; and then show that $C - (\bigcup_{C' \in N} C')$ is nonempty for any subfamily N of maximal cliques with $|N| \leqslant 4r$ and $C \notin N$ (Lemma 4.4.10).

We first claim that $\mu_x = r$ is equivalent to the emptiness of the set $M_s := \{C | \forall x \in C, \mu_x > r\}$ by Theorem 4.3.1 and Lemma 4.4.9, we then claim that M_s is empty in terms of the principle of inclusion and exclusion and some inequalities by contradictory argument over he conditions $|N| \leqslant 4r$ and the fact that each point x with $\mu_x > r$ lies in at least five cliques.

Suppose that there exists an integer a such that Theorem 4.3.1 (a,b,c) and $|C| \geqslant k - a\mu$ for each $C \in M$, and $m \leqslant a \leqslant 2m - 1$ are fulfilled.

Let $m = r$, $n = k + r + c - 1 - t$, $\mu = rt$, $\gamma = rc$, $R = r(k-1)$. Assume that $k > rt(4a + r + 5)$, $k > (c+1)t$, $r \geqslant 3$ and $r(c + r - 1) \leqslant (r-1)t$.

The relation of Lemma 4.4.2 $\sim$ Lemma 4.4.11:

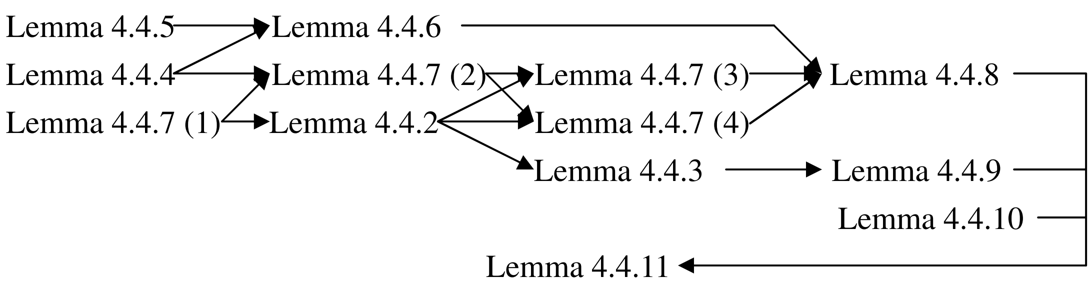

Figure 3: Section 3 in [7], Metsch

Lemma 4.4.2 ([7], Lemma 3.10) $l := \mu_x$, $x \in C_1, C_2, \ldots, C_l$. Then

(1) $\sum_{i=1}^{l}(|C_i|-1) \leqslant r(k-1)+\frac{1}{2}\gamma$;

(2) $\sum_{i=1}^{l}(|C_i|-1) = r(k-1)$ if $l=r$.

From $\sum_{i=1}^{l}(|C_i|-1) = r(k-1)$, we have:

Lemma 4.4.3 ([7], Lemma 3.12) If $\mu_x = r$ and $x \in C$, then $|C| \geqslant k-(r-1)(c+1)$.

Lemma 4.4.4 ([7], Lemma 3.11) $\forall C \in M$, $|C| < k+c+1$.

Lemma 4.4.5 ([7], Lemma 3.13) $\sum_{C \neq C' \in M} |C \cap C'|^2 \leqslant v(r-1)t + 3r(c+1)\sum_{x \in V}(\mu_x - r)$.

Lemma 4.4.6 ([7], Lemma 3.14) $\sum_{x \in V}(s_x - 3rtk(\mu_x - r)) \leqslant vt(r-1)(k+c+1)$.

Lemma 4.4.7 Let $l := \mu_x$, $x \in C_1, C_2, \ldots, C_l$. Let $z_i := |C_i| - 1$, $i = 1, 2, \ldots, l$, and $z := \max\{z_i | i = 1, 2, \ldots, l\}$. Then

(1) $l \cdot s_x \geqslant (n-m)(R+\gamma) + R\mu - 2(1+l-m)z\gamma - \sum_{i=1}^{l} z_i^2$ ([7], Lemma 3.15);

(2) $l \cdot s_x \geqslant r(k-1)^2 + r(r-1)tk - r(r-1)t - rtc - r(c+1)^2 - 2\gamma(k+c)(l-r) - \sum_{i=1}^{l} z_i^2$ ([7], Lemma 3.16);

(3) $s_x \geqslant (r-1)tk - (r-1)t^2$ for every vertex x satisfying $\mu_x = r$ ([7], Lemma 3.17); and

(4) $\mu_x s_x \geqslant r(r-1)tk + k(k-4\mu-a\mu)(\mu_x - r)$ for every vertex x satisfying $\mu_x > r$ ([7], Lemma 3.18).

Lemma 4.4.8 ([7], Lemma 3.19) There exist at most $2rk$ vertices x satisfying $\mu_x > r$, i.e., $v_s \leqslant 2rk$.

Lemma 4.4.9 ([7], Lemma 3.20) Every vertex x satisfying $\mu_x > r$ lies in at least five cliques, which contain only vertices y satisfying $\mu_y > r$.

Lemma 4.4.10 ([7], Lemma 3.21)

(1) $|C_1 \cap C_2| \leqslant \mu$ for $C_1 \neq C_2$.

(2) $C \in M$, $N \subseteq M$ with $|N| \leqslant 4r$ and $C \notin N$, then there exist $x \in C$ and $x \notin C' \in N$ for all C'.

Lemma 4.4.11 ([7], Lemma 3.22) $\mu_x = r$ for each vertex x.

5 A Class of Strongly Regular Multigraphs

In this section, we will use the special properties of the definition of alternating form graph to define a symmetric association scheme. From the symmetric association scheme, we have that the alternating form graph is a distance regular graph. At last, we will define a graph Γ which is the induced subgraph of the alternating form graph, and give the multiplicity on the edges, and we will get a class of strongly regular multigraphs.

Definition 5.1 The *alternating form graph* $Alt(n, q)$ is the simple graph with vertex set $V = \{A | A \in M_{n\times n}(GF(q)), A = -A^T\}$ and the edge set $E = \{(A, B) | A, B \in V, rank(A - B) = 2\}$.

Since the alternating form graph $Alt(n, q)$ is defined on the set of all skew-symmetric $n \times n$ matrices over $GF(q)$, the rank of $A - B$ is $2i$ for any two matrices A and B in $Alt(n, q)$. Let $R_i = \{(A, B) | A, B \in Alt(n, q), rank(A - B) = 2i\}$, the relation classes

$\{R_0, R_1, \ldots, R_d\}$, defined on $Alt(n,q) \times Alt(n,q)$, where $d = \lceil \frac{n}{2} \rceil$; then

(1) $(Alt(n,q), (R_i)_{i=0}^d)$ is a symmetric association scheme.

(2) $(Alt(n,q), R_1)$ is a *distance regular graph* with the intersection numbers $\{b_0, b_1, \ldots,$ $b_{d-1}; c_0, c_1, \ldots, c_d\}$.

Theorem 5.2 Each maximal cliques of the alternating form graph $Alt(n,q)$ is either isomorphic to $\begin{pmatrix} 0 & x_2 & x_3 & \ldots & x_n \\ -x_2 & 0 & 0 & \ldots & 0 \\ -x_3 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_n & 0 & 0 & \ldots & 0 \end{pmatrix}$ or to $\begin{pmatrix} 0 & x & y & 0 & \cdots & 0 \\ -x & 0 & z & 0 & \cdots & 0 \\ -y & -z & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$ respectively, called typed I and type II respectively.

For studying possible combinatorial geometric structures over $Alt(n,q)$, we shall study the *matrix representations* of those maximal cliques of both types containing the zero form. The others may be obtained simply by translation. Without loss of generality, we may assume that V is an inner product space with a fixed orthonormal basis $\{v_1, v_2, \ldots, v_n\}$. If $v \in V$ is a nonzero vector, then $< v >=< \alpha v >$ for all $\alpha \in GF^*(q)$, we may assume that $v = \sum_{1 \leqslant j \leqslant k-1} \alpha_j v_j + v_k$ for some k and $\alpha_j \in GF(q)$. Consider the nest

$$V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n (= V)$$

of subspaces of V, where $V_i =< v_1, v_2, \ldots, v_i >$, $1 \leqslant i \leqslant n$, and in particular V_0 is the trivial subspace of V. Since $\dim V_i$ is i, V_i has $(q^i - 1)/(q-1)$ one-dimensional subspaces, they are

$$< \sum_{1 \leqslant j \leqslant k-1} \alpha_j v_j + v_k >$$

where $\alpha_j \in GF(q)$, $1 \leqslant j \leqslant k-1$. Their corresponding perpendicular subspaces (i.e.

hyperplanes) are

$$< \sum_{1\leqslant j\leqslant k-1} \alpha_j v_j + v_k >^{\perp} = < v_{k+1}, v_{k+2}, \ldots, v_n, v_j - \alpha_j v_k | 1 \leqslant j \leqslant k-1 > .$$

Proposition 5.3 Show that each of these hyperplanes uniquely determines a maximal clique of type I of the zero form.

Proof: When $k = 1$, $v = v_1$ and $< v >^{\perp} = < v_2, v_3, \ldots, v_n >$, it follows that its matrix representation is

$$l_v (= l_\infty) = \{ \begin{pmatrix} 0 & x_2 & x_3 & \ldots & x_{n-1} & x_n \\ -x_2 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} | x_i \in GF(q) \}$$

consisting of those matrices whose first two rows as shown above, their first two columns obtained by skew-symmetry, and zero all other entries. Similar convention is used in the following.

When $k \geqslant 2$, the clique determined by $v = \sum_{1\leqslant j\leqslant k-1}(\alpha_j v_j + v_k)$ or its perpendicular space is denoted by $l_v = l_{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}}$ if there is no confusion. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis of $GF(q)^n$, i.e., $e_i = (0, 0, \ldots, 0, 1, 0, 0, \ldots, 0)$ with 1 in the i-th entry, $1 \leqslant i \leqslant n$.

Let

$$L_i = \{ e_i^T \cdot x - x^T \cdot e_i | x \in GF(q)^n \text{ with 0 in its } i\text{-th entry} \},$$

and,

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 1, 0, 0, \ldots, 0) \in GF(q)^n$$

and $P_i(\alpha)$ denote the matrix obtained from the identity matrix by replacing its i-th row by the vector α. Then the matrix representation of $l_v = l_{\alpha_1,\alpha_2,\ldots,\alpha_{i-1}}$, with respect to the fixed base $\{v_1, v_2, \ldots, v_n\}$, is given by

$$\begin{aligned} l_v &= l_{\alpha_1,\alpha_2,\ldots,\alpha_{i-1}} \\ &= P_i(\alpha)^T \cdot L_i \cdot P_i(\alpha) \\ &= \{P_i(\alpha)^T \cdot (e_i \cdot x - x^T \cdot e_i) \cdot P_i(\alpha) | x \in GF(q)^n \text{ with 0 in its } i\text{-th entry}\} \\ &= \{\alpha^T \cdot x - x^T \cdot \alpha | x \in GF(q)^n \text{ with 0 in its } i\text{-th entry}\}. \end{aligned}$$

Remark For attenuated space, i.e., $M_{k\times n}(GF(q))$, those blocks of the zero matrix can be expressed as $\{\alpha^T \cdot x | x \in GF(q)^n\}$ where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 1, 0, 0, \ldots, 0) \in GF(q)^n$ for all nonzero $\alpha \in GF(q)^n$ with 1 in its last nonzero entry.

For each i, there are q^{i-1} vectors of the form

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 1, 0, 0, \ldots, 0) \in GF(q)^n,$$

and each such vector α uniquely determines a maximal clique $l_v = l_{\alpha_1,\alpha_2,\ldots,\alpha_{i-1}} = P_i(\alpha)^T \cdot L_i \cdot P_i(\alpha)$ of zero form. Hence all maximal cliques of type I of the zero form are obtained in this way. The above observation can be summarized as follows:

Proposition 5.4

(1) Each maximal clique of type I consists of q^{n-1} vertices.

(2) Each vertex is in exactly $\frac{q^n-1}{q-1}$ maximal cliques of type I.

Proof: The maximal clique of type I consists of q^{n-1} vertices. since all maximal cliques of type I is isomorphic to $\begin{pmatrix} 0 & x_2 & x_3 & \ldots & x_n \\ -x_2 & 0 & 0 & \ldots & 0 \\ -x_3 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_n & 0 & 0 & \ldots & 0 \end{pmatrix}$. And the maximal clique is uniquely determined by hyperplane H with $\dim(H) = n-1$. Then the number of hyperplanes is $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q = \frac{q^n-1}{q-1}$. Q.E.D.

Proposition 5.5 Those $(q^n-1)/(q-1)$ maximal cliques of type I containing the zero form can be expressed as

$$\{\alpha^T \cdot x - x^T \cdot \alpha | x = (x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, x_{i+2}, \ldots, x_n), x_j \in GF(q)^n\},$$

for $\alpha = (1, 0, 0, \ldots, 0)$ or $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, 1, 0, 0, \ldots, 0)$ where $i \geqslant 2$ and $\alpha_j \in GF(q)^n$.

Now, we turn to the intersection properties among those maximal cliques in the distance regular graph $Alt(n, q)$.

Lemma 5.6 Let C_1 and C_2 be two maximal cliques of type I, then $|C_1 \cap C_2| = 0$ or q. Moreover, if $C_1 \cap C_2 \neq \phi$, then $C_1 \cap C_2$ is isomorphic to

$$\begin{pmatrix} 0 & x & 0 & 0 & \ldots & 0 \\ -x & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

and there exists another $q-1$ maximal cliques C_3, C_4, $\ldots$, C_{q+1} of type I such that $\sum_{i=1}^{q+1} C_i = C_1 \cap C_2$.

Proof: Clearly, $|C_1 \cap C_2| = 0$ or q. C_1 and C_2 are determine by hyperplanes H_1 and H_2, respectively, with $\dim(H_i) = n-1$, and $\dim(H_1 \cap H_2) = n-2$ for $i = 1, 2$. In addition to H_1, H_2, there are another $q-1$ hyperplanes containing $H_1 \cap H_2$. They determine the rest $q-1$ maximal cliques of type I are required. Q.E.D.

Let $\mathcal{B}$ be the set of all maximal cliques of type I in $Alt(n, q)$. Then $\pi = (A_n, \mathcal{B}, \in)$ turns out to be an incidence structure with some intersecting properties. Elements in $\mathcal{B}$ are called blocks of this incidence structure. The following proposition is simply a restatement of proposition 5.4 and lemma 5.6.

Proposition 5.7

(1) Each block B consists of q^{n-1} points,

(2) Each point is incident with exactly $(q^n - 1)/(q - 1)$ blocks,

(3) Any two distinct blocks are incident with either 0 or q common points,

(4) If B_1, $B_2 \in \mathcal{B}$ are distinct and $B_1 \cap B_2 \neq \phi$ then there are another $q - 1$ blocks, say B_3, B_4, …, B_{q+1}, such that $\bigcap_{i=1}^{q+1} B_i = B_1 \cap B_2$ consists of q points.

Definition 5.8 A *singular line* is the intersection of two distinct maximal cliques.

When we only consider the maximal cliques of type I, we have the constant size of cliques. And if we define the graph with multiedge by $m_{A,B}$ = the number of singular lines containing vertices A and B, then we have $m_{A,B} = 0$, 1 or $q + 1$.

Consider the definition of strongly regular multigraph, $\gamma = \sum_{y \in V} m_{x,y}(m_{x,y} - 1)$. For each A and B lie in the same maximal cliques, there are exactly 2 entries of B different from those of A. So we have $\gamma = \sum_B (q + 1)q$ is the constant.

Definition 5.9 For a fixed matrix $M'_{(n-2)\times(n-2)}(GF(q))$ with odd q, let Γ be a multigraph with defined on (V, E) with $\{M \mid M$ is a skew-symmetric matrix over $GF(q)$ satisfying $M = \begin{pmatrix} X & Y \\ -Y^T & M' \end{pmatrix}\}$ as the vertex set, and for any two vertices A, B, $A \sim B$ if and only if $rank(A - B) = 2$. Moreover, let $m_{A,B}$ be the number of singular line containing vertices A and B.

Theorem 5.10 Γ is a strongly regular multigraph with parameters

$$(m', n', \mu', \gamma', R') = (q + 1, q^{n-1}, q^2(q + 1), q(q^2 - 1), (q + 1)(q^{n-1} - 1)),$$

and Γ is the collinearity graph of a $1\frac{1}{2}$-design with parameters

$$(r, k, t, c) = (q + 1, q^{n-1}, q^2, q(q - 1)).$$

Proof:

For a fixed vertex $A \in V$,

(1) $m' = q + 1$ is the number of maximal cliques contains A;

(2) there exists exactly $q + 1$ cliques containing A and B for adjacent vertices A and B, and each maximal clique has size q^{n-1}; thus $R' = (q + 1)(q^{n-1} - 1)$.

(3) $\gamma' = \sum_{B\sim A} m_{A,B}(m_{A,B} - 1) = \sum_{B\sim A}(q + 1)q = (q - 1)q(q + 1) = q(q^2 - 1)$, since $m_{A,B} = 1$ or 0 whenever B does not lie in the $q+1$ maximal cliques which contains A.

(4) since $m'(m' - n') + \mu' = R' + \gamma'$, then $(q + 1)(n' - (q + 1)) + \mu' =(q + 1)(q^{n-1} - 1) + q(q^2 - 1)$, it follows that $\mu' = (q + 1)(q^{n-1} - n' + q^2)$.

(5) since $\mu' v = (R' + m')(R' + m' - n')$, then $\mu' q^{2n-3} = ((q + 1)(q^{n-1} - 1)) + (q + 1)((q+1)(q^{n-1} - 1) + (q+1) - n')$. It follows that $\mu' q^{n-2} = (q+1)((q+1)q^{n-1} - n')$;

combine (4) and (5), we have $\mu' q^{n-2} = q^{n-2}(q + 1)(q^{n-1} - n' + q^2)$. Hence $n' = q^{n-1}$ and $\mu' = (q + 1)(q^{n-1} - n' + q^2) = q^2(q + 1)$.

We then have $(r, k, t, c) = (q + 1, q^{n-1}, q^2, q(q - 1))$ as required. Q.E.D.

The combinatorial interpretations of the parameters (r, k, t, c) of the $1\frac{1}{2}$-design under consideration are given below:

(1) $r =$ number of cliques contains a fixed vertex $= q + 1$.

(2) $k =$ the clique size $= q^{n-1}$.

(3) Fixed $y \notin B$ where y is a point and B is a block, then $t = \sum_{x\in B} m_{x,y} = q^2$.

(4) Fixed $y \in B$, $m_{x,y} - 1 = q$ or 0 by $x \in B$ and

a. $m_{x,y} = 1$ if x does not lie in the $q + 1$ blocks containing y,

b. $m_{x,y} = q + 1$ if x lies in the $q + 1$ blocks containing y, thus $c = \sum_{x \in B-\{y\}} (m_{x,y} - 1) = q(q-1)$.

We constructed the strongly regular multigraph above. Now, we are interesting that in what conditions, we can make sure the $1\frac{1}{2}$-design with parameters

$$(r, k, t, c) = (q+1, q^{n-1}, q^2, q(q-1))$$

is the *unique* incidence structure such that the collinearity graph is this strongly regular multigraph with parameters

$$(m', n', \mu', \gamma', R') = (q+1, q^{n-1}, q^2(q+1), q(q^2-1), (q+1)(q^{n-1}-1)).$$

We now check the numerical constraints required in Theorem 4.2.17 ([9], Theorem 4.4) for the uniqueness of the corresponding incidence structure with respect to the strongly regular multigraph under consideration; i.e., to find conditions to guarantee that $m' \geqslant 2$, integral $\mu' \equiv 0 \bmod m'$, $\mu' > 0$, and $n' > \max\{m' - 1 + \frac{(\mu'+m')\gamma'}{m'^2}, 2(m'-1)(\mu'+1-m') + 2\gamma', \frac{m'(m'-1)}{2}(\mu'+1) + m'\frac{\gamma'}{2} + m' - 1\}$.

Theorem 5.11 The strongly regular multigraph Γ is the point graph of a unique $1\frac{1}{2}$-design whenever $n = 6$, $q \geqslant 4$ or $n \geqslant 7$, $q \geqslant 3$.

Proof: Clearly, $\mu' = q^2(q+1) \equiv 0 \bmod m'(= q+1)$ and $\mu' > 0$, $m' \geqslant 2$.

(1) Since $q^{n-1} > q + (q^2+1)q(q-1) = q^4 - q^3 + q^2$, then

$$q^{n-1} > (q+1) - 1 + \frac{(q^2(q+1) + (q+1))(q(q^2-1))}{(q+1)^2},$$

i.e., the condition $n' > m' - 1 + \frac{(\mu'+m')\gamma'}{m'^2}$ hold.

(2) Since $q^{n-1} > 2q((q^2-1)(q+1)+1) + 2q(q^2-1) = 2q^4 + 4q^3 - 2q^2 - 2q$, then

$$q^{n-1} > 2((q+1)-1)(q^2(q+1) + 1 - (q+1)) + 2q(q^2-1),$$

i.e., the condition $n' > 2(m'-1)(\mu'+1-m') + 2\gamma'$ hold.

(3) Since $q^{n-1} > \frac{q(q+1)}{2}(q^3+q^2+1) + (q+1)\frac{q(q^2-1)}{2} + q = \frac{1}{2}(q^5+3q^4+2q^3+2q)$, then

$$q^{n-1} > \frac{(q+1)((q+1)-1)}{2}(q^2(q+1)+1) + (q+1)\frac{q(q^2-1)}{2} + (q+1) - 1,$$

i.e., the condition $n' > \frac{m'(m'-1)}{2}(\mu'+1) + m'\frac{\gamma'}{2} + m' - 1$ hold.

Combine (1) $\sim$ (3), we have

$$q^{n-1} > \max\{q^4 - q^3 + q^2, 2q^4 + 4q^3 - 2q^2 - 2q, \frac{1}{2}(q^5 + 3q^4 + 2q^3 + 2q)\} \qquad (*)$$

(*) holds if $n = 6$, $q \geqslant 4$ and if $n \geqslant 7$, $q \geqslant 3$. Q.E.D.

From Theorem 5.10 and 5.11, we have:

Theorem 5.12 If Γ is a strongly regular multigraph with parameters $(m', n', \mu', \gamma', R') = (q+1, q^{n-1}, q^2(q+1), q(q^2-1), (q+1)(q^{n-1}-1))$ with odd prime q and integer n such that $n = 6$, $q \geqslant 4$ or $n \geqslant 7$, $q \geqslant 3$, then Γ is the collinearity graph of a unique $1\frac{1}{2}$-design with parameters $(r, k, t, c) = (q+1, q^{n-1}, q^2, q(q-1))$.

References

[1] R. C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, Pacific J. Math. 13 (1963) 389-419.

[2] R. C. Bose, S. S. Shrikhande and N. M. Singhi, Edge regular multigraphs and partial geometric designs with an application to the embedding of quasi-residual designs, Coll. Int. Sul. Teo. Combin, Tom I (Acc Naz. Lincei, Roma, 1976) 49-81.

[3] Chris Godsil, Gordon Royle, Algebraic Graph Theory, Springer, 2001.

[4] W. H. Haemers, Matrix techniques for strongly regular graphs and related geometries, 2000, Lecture notes for the Intensive Course on Finite Geometry and its Applications.

[5] J. F. Lawless, Block intersections in quasi-residual designs, Aequationes Math. 5 (1970) 40-46.

[6] K. Metsch, Improvement of Bruck's completion theorem, Designs Codes Cryptography 1 (1991) 99-116.

[7] K. Metsch, Quasi-residual designs, $1\frac{1}{2}$-designs, and strongly regular multigraphs, Discrete Mathematics 143 (1995) 167-188.

[8] A. Neumaier, $t\frac{1}{2}$-designs, J. Comb. Theory Ser. A 28 (1980) 226-248.

[9] A. Neumaier, Quasi-residual designs, $1\frac{1}{2}$-designs, and strongly regular multigraphs, Geom. Dedicata 12 (1982) 351-366.

[10] S. S. Shrikhande, Relations between incomplete block designs, Contributions to Probability and Statistics. Essays in honor of Harold Hotelling, Stanford U. Press, 388-395.

[11] N. M. Singhi and S. S. Shrikhande, Embedding of quasi-residual designs with $\lambda = 3$, Utilitas Mathematica 4 (1973) 35-53.