Mutually orthogonal hamiltonian connected graphs

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In this work, we concentrate on those \( n \)-vertex graphs \( G \) with \( n \geq 4 \) and \( \bar{v} \leq n - 4 \). Let \( P_1 = \langle u_1, u_2, \ldots, u_n \rangle \) and \( P_2 = \langle v_1, v_2, \ldots, v_n \rangle \) be any two hamiltonian paths of \( G \). We say that \( P_1 \) and \( P_2 \) are orthogonal if \( u_1 = v_1, u_n = v_n, \) and \( u_i \neq v_i \) for \( q \in \{2, n - 1\} \). We say that a set of hamiltonian paths \( \{P_1, P_2, \ldots, P_k\} \) of \( G \) are mutually orthogonal if any two distinct paths in the set are orthogonal. We will prove that there are at least two orthogonal hamiltonian paths of \( G \) between any two different vertices. Furthermore, we classify the cases such that there are exactly two orthogonal hamiltonian paths of \( G \) between any two different vertices. Aside from these special cases, there are at least three mutually orthogonal hamiltonian paths of \( G \) between any two different vertices.

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1. Introduction

In this work, a network is represented as a loopless undirected graph. For graph definitions and notation we follow [1]. \( G = (V, E) \) is a graph if \( V \) is a finite set and \( E \) is a subset of \( \{(u, v) \mid (u, v) \) is an unordered pair of \( V \} \). We say that \( V \) is the vertex set and \( E \) is the edge set. Two vertices \( u \) and \( v \) are adjacent if \( u, v \in E \). Let \( S \) be a subset of \( V \). The subgraph of \( G \) induced by \( S \) is the graph \( G[S] \) with \( V(G[S]) = S \) and \( E(G[S]) = \{(u, v) \mid (u, v) \in E, u, v \in S\} \). The complement \( \bar{G} \) of a graph \( G \) is with the same vertex set \( V(G) \) defined by \( u, v \in E(G) \) if and only if \( u, v \not\in E(G) \). We use \( \bar{E} \) to denote \( |E(G)| \).

The degree of a vertex \( u \) of \( G \), \( \deg_G(u) \), is the number of edges incident with \( u \). A path, \( \langle v_0, v_1, v_2, \ldots, v_k \rangle \), is an ordered list of distinct vertices such that \( v_i \) and \( v_{i+1} \) are adjacent for \( 0 \leq i \leq k - 1 \). A path is a hamiltonian path if its vertices are distinct and span \( V \). A graph \( G \) is hamiltonian connected if there exists a hamiltonian path joining any two vertices of \( G \). A cycle, \( \langle v_0, v_1, \ldots, v_k, v_0 \rangle \), is a path with at least three vertices such that the first vertex is the same as the last vertex. A cycle is a hamiltonian cycle if it traverses every vertex of \( G \) exactly once. A graph is hamiltonian if it has a hamiltonian cycle.

Let \( P_1 = \langle u_1, u_2, \ldots, u_n \rangle \) and \( P_2 = \langle v_1, v_2, \ldots, v_n \rangle \) be any two hamiltonian paths of an \( n \)-vertex hamiltonian connected graph \( G \). We say that \( P_1 \) and \( P_2 \) are orthogonal if \( u_1 = v_1, u_n = v_n, \) and \( u_i \neq v_i \) for \( q \in \{2, n - 1\} \). We say that a set of hamiltonian paths \( \{P_1, P_2, \ldots, P_k\} \) of \( G \) are mutually orthogonal if any two distinct paths in the set are orthogonal.

In this work, we concentrate on those \( n \)-vertex graphs \( G \) with \( n \geq 4 \) and \( \bar{v} \leq n - 4 \). By the famous Ore's Theorem [2], \( G \) is hamiltonian connected. Yet, we will prove that there are at least two orthogonal hamiltonian paths of \( G \) between any two different vertices. Furthermore, we classify the cases such that there are exactly two orthogonal hamiltonian paths of \( G \) between any two different vertices. Thus, there are at least three mutually orthogonal hamiltonian paths of \( G \) between any two different vertices except for the cases mentioned above. This result can be used to compute the fault-tolerant hamiltonian connectivity of the WK-recursive networks [3].

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2. Mutually orthogonal hamiltonian paths

The following theorem is proved by Ore [2].

**Theorem 1** ([2]). Assume that $G$ is an $n$-vertex graph with $n \geq 4$. Then $G$ is hamiltonian if $\bar{d} \leq n - 3$, and is hamiltonian connected if $\bar{d} \leq n - 4$.

Let $G$ and $H$ be two graphs. We use $G + H$ to denote the disjoint union of $G$ and $H$. We use $G \vee H$ to denote the graph obtained from $G + H$ by joining each vertex of $G$ to each vertex of $H$. For $1 \leq m < n/2$, let $C_{m,n}$ be the graph $(K_m + K_{n-2m}) \vee K_m$. See Fig. 1 for an illustration. The following theorem is proved by Chvátal [4].

**Theorem 2** ([4]). If $G$ is an $n$-vertex graph where $n \geq 3$ and $|E(G)| > C_{n-1}^2 + 1$, then $G$ is hamiltonian. Moreover, the only non-hamiltonian graphs with $n$ vertices and $C_{n-1}^2 + 1$ edges are $C_{1,n}$ and, for $n = 5, C_{2,5}$.

Suppose that $G$ is an $n$-vertex graph with $\bar{d} \leq n - 4$. Assume that $n = 4$. Obviously, $G$ is isomorphic to $K_4$. It is easy to check that there are exactly two orthogonal hamiltonian paths between any two distinct vertices of $G$.

Assume that $n = 5$. Obviously, $G$ is isomorphic either to $K_5$ or to $K_5 - e$ where $e$ is any edge of $K_5$. We label the vertices of $K_5$ with $\{1, 2, 3, 4, 5\}$ and we set $e = (1, 2)$. Suppose that $G$ is isomorphic to $K_5$. It is easy to check that there are exactly three mutually orthogonal hamiltonian paths of $G$ between any two vertices. Suppose that $G$ is isomorphic to $K_5 - (1, 2)$. By brute force, we can check that there are exactly three mutually orthogonal hamiltonian paths between vertices 1 and 2. However, there are exactly two orthogonal hamiltonian paths between the remaining pairs.

Now, we assume that $n \geq 6$. Let $s$ and $t$ be any two distinct vertices of $G$. Let $H$ be the subgraph of $G$ induced by the remaining $(n - 2)$ vertices of $G$. We have the following two cases:

**Case 1:** $H$ is hamiltonian. We can label the vertices of $H$ with $\{0, 1, 2, \ldots, n - 3\}$ such that $\langle 0, 1, 2, \ldots, n - 3, 0 \rangle$ forms a hamiltonian walk in $H$. We use the notation $[i]$ to denote $i \mod (n - 2)$. Let $Q$ denote the set $\{i \mid (s, [i + 1]) \in E(G) \text{ and } (i, t) \in E(G)\}$. Since $\bar{d} \leq n - 4$, $|Q| \geq n - 2 - (n - 4) = 2$. There are at least two elements $q_1, q_2$ in $Q$. We set $P_1$ as $\langle s, [q_1 + 1], [q_1 + 2], \ldots, [q_1] \rangle$, for $j = 1, 2$. Then $P_1$ and $P_2$ are two orthogonal hamiltonian paths between $s$ and $t$.

Suppose that $\bar{d} \leq n - 5$, $(s, t) \notin E$, or $H$ is not isomorphic to the complete graph $K_{n-2}$. Then $|Q| \geq 3$. Let $q_1, q_2,$ and $q_3$ be the three elements in $Q$. For $j = 1, 2,$ and $3$, we set $P_1$ as $\langle s, [q_j + 1], [q_j + 2], \ldots, [q_j] \rangle$. Then $P_1, P_2$, and $P_3$ are three mutually orthogonal hamiltonian paths between $s$ and $t$.

Thus, we consider $\bar{d} = n - 4$, $(s, t) \in E$, and $H$ is isomorphic to the complete graph $K_{n-2}$. Let $ST$ be the set of vertices in $H$ that are adjacent to $s$ and $t$, let $S$ be the set of vertices in $H$ that are adjacent to $s$ but not adjacent to $t$, let $T$ be the set of vertices in $H$ that are not adjacent to $s$ but adjacent to $t$, and let $ST$ be the set of vertices in $H$ that are neither adjacent to $s$ nor adjacent to $t$.

Let $a = |ST|, b = |ST|, c = |ST|, \text{ and } d = |ST|$. Without loss of generality, we assume that $\deg_c(s) \geq \deg_c(t)$. Then $b \geq c, b + c + 2d = n - 4$, and $a + b + c + d = n - 2$. Thus, $a - d = 2$. Hence, $a \geq 2$.

Suppose $a \geq 3$. Let $q_1, q_2,$ and $q_3$ be three vertices in $ST$ and $q_4, q_5, \ldots, q_{n-2}$ be the remaining vertices of $H$. We set $P_1$ as $\langle s, q_1, q_2, X, q_3, t \rangle$, $P_2$ as $\langle s, q_2, q_3, Y, q_1, t \rangle$, and $P_3$ as $\langle s, q_3, Z, q_1, q_2, t \rangle$ where $X, Y,$ and $Z$ are any permutations of $q_4, q_5, \ldots, q_{n-2}$. Obviously, $P_1, P_2,$ and $P_3$ are three mutually orthogonal hamiltonian paths between $s$ and $t$.

Suppose $a = 2$. Then $d = 0$. Suppose $c \geq 1$. Then $b \geq 1$. We rearrange the vertices of $H$ so that 0 is a vertex in $ST$, 1 and 2 are the vertices in $ST$, 3 is a vertex in $ST$, and 4, 5, $\ldots, n - 3$ are the remaining vertices. Obviously, $\langle 0, 1, 2, \ldots, n - 3, 0 \rangle$ forms a hamiltonian cycle of $H$. Let $Q$ denote the set $\{i \mid (s, [i]) \in E(G) \text{ and } ([i + 1, t] \in E(G)\}$. Obviously, $|Q| \geq 3$. Thus, there are three mutually orthogonal hamiltonian paths between $s$ and $t$.

Finally, we consider $a = 2, d = 0, \text{ and } c = 0$. Thus, $b = n - 4$. In this case, $s$ is adjacent to $t$ and all the vertices in $H$; $t$ is adjacent to $s$ and exactly two vertices in $H$, say $q_1$ and $q_2$. Let $\langle s = v_1, v_2, v_3, \ldots, v_n = t \rangle$ be a hamiltonian path of $G$ between $s$ and $t$. Obviously, $v_{n-1}$ is either $q_1$ or $q_2$. Therefore, there are exactly two orthogonal hamiltonian paths between $s$ and $t$.

**Case 2:** $H$ is non-hamiltonian. There are exactly $(n - 2)$ vertices in $H$. By Theorem 2, there are exactly $(n - 4)$ edges in the complement of $H$ and $H$ is isomorphic to $C_{1,n-2}$ or $C_{2,5}$. Hence, $s$ is adjacent to $V(G) - \{s\}$ and $t$ is adjacent to $V(G) - \{t\}$. 

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**Fig. 1.** Illustration of $C_{m,n}$. 

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We can construct two orthogonal hamiltonian paths of $G$ between $s$ and $t$ as the following cases:

**Subcase 2.1:** $H$ is isomorphic to $C_{2,5}$. We label the vertices of $C_{2,5}$ with $\{1, 2, 3, 4, 5\}$ as shown in Fig. 2(a). Let $P_1 = (s, 1, 2, 3, 4, 5, t)$ and $P_2 = (s, 3, 4, 5, 2, 1, t)$. Then $P_1$ and $P_2$ form the required orthogonal paths. By brute force, we can check that there are exactly two orthogonal hamiltonian paths between $s$ and $t$.

**Subcase 2.2:** $H$ is isomorphic to $C_{1,n-2}$. We label the vertices of $C_{1,n-2}$ with $\{1, 2, \ldots, n - 2\}$ as shown in Fig. 2(b). Let $P_1 = (s, 1, 2, 3, \ldots, n - 2, t)$ and $P_2 = (s, 3, 4, \ldots, n - 2, 2, 1, t)$. Then $P_1$ and $P_2$ form the orthogonal hamiltonian paths. Let $(s = v_1, v_2, \ldots, v_n = t)$ be any hamiltonian path of $G$ between $s$ and $t$. Obviously, 1 is either $v_2$ or $v_{n-1}$. Therefore, there are exactly two orthogonal hamiltonian paths between $s$ and $t$.

From the above discussions, we have the following theorem.

**Theorem 3.** Assume that $G$ is an $n$-vertex graph with $n \geq 4$ and $\bar{e} \leq n - 4$. Let $s$ and $t$ be any two vertices of $G$. Then there are at least two orthogonal hamiltonian paths of $G$ between $s$ and $t$. Moreover, there are at least three mutually orthogonal hamiltonian paths of $G$ between $s$ and $t$ except for the following cases:

1. $G$ is isomorphic to $K_4$ where $s$ and $t$ are any two vertices of $G$.
2. $G$ is isomorphic to $K_5 - \{1, 2\}$ where $s$ and $t$ are any two vertices except for $\{s, t\} = \{1, 2\}$.
3. The subgraph $H$ induced by $V(G) - \{s, t\}$ is a complete graph with $n \geq 6$ where $s$ is adjacent to $t$ and all the vertices in $H$ and $t$ is adjacent to $s$ and exactly two vertices in $H$.
4. The subgraph induced by $V(G) - \{s, t\}$ is isomorphic to $C_{2,5}$ where $s$ is adjacent to $V(G) - \{s\}$ and $t$ is adjacent to $V(G) - \{t\}$.
5. The subgraph induced by $V(G) - \{s, t\}$ is isomorphic to $C_{1,n-2}$ with $n \geq 6$ where $s$ is adjacent to $V(G) - \{s\}$ and $t$ is adjacent to $V(G) - \{t\}$.

**References**