THE $\beta$-ASSIGNMENT PROBLEM IN GENERAL GRAPHS

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Scope and Purpose—The purpose of this paper is to study a variant assignment problem in operations research. Suppose we have a set of jobs and a set of workers. The qualifications of workers for each job are known in advance. The problem is to assign the jobs to the workers so that the maximum number of jobs a worker has to perform is minimized. If there are pairs of jobs that can help each other, we can assign them in a symbiotic pairs without assigning them to workers. The fact that two jobs can help each other is also a known. This paper studies this general problem from an algorithmic point of view.

Abstract—We study a variation of the assignment problem in operations research and formulate it in terms of graphs as follows. Suppose $G=(V,E)$ is a graph and $U$ a subset of $V$. A $\beta$-assignment of $G$ with respect to $U$ is an edge set $X$ such that $\deg_x(v)=1$ for all vertices $v$ in $U$, where $\deg_x(v)$ is the degree of $v$ in the subgraph of $G$ induced by the edge set $X$. The $\beta$-assignment problem is to find a $\beta$-assignment $X$ such that $\beta(X)=\max\{\deg_x(v):v\in V-U\}$ is minimum. The purpose of this paper is to give an $O(n^3)$-time algorithm for the $\beta$-assignment problem in general graphs. As byproducts, we also obtain a duality theorem as well as a necessary and sufficient condition for the existence of a $\beta$-assignment for a general graph. The latter result is a generalization of Tutte's theorem for the existence of a perfect matching of a general graph. © 1997 Elsevier Science Ltd

1. INTRODUCTION

The assignment problem is a well-known subject in operations research [1–5]. Chang and Lee [6] considered the following variation of the assignment problem. There is a set $S$ of $n$ jobs and a set of $T$ of $m$ workers. A worker's qualifications for a given job are known. The problem is to assign jobs to workers so that the maximum number of jobs a worker has to perform is optimized. The problem was formulated as follows in terms of bipartite graphs. Consider the bipartite graph $G=(S,T,E)$ in which $(S,T)$ is a bipartition of the vertex set, and $(i,j)\in E$ if and only if worker $j$ is qualified for job $i$. The problem is then to find an edge subset $X$ of $E$ such that $\deg_x(v)=1$ for all vertices $v$ in $S$ and $\max\{\deg_x(v):v\in T\}$ is as small as possible, where $\deg_x(v)$ is the degree of $v$ in the subgraph of $G$ induced by $X$.

This paper studies the following variation of the above assignment problem. In this variation, there are pairs of jobs that can help each other so that we do not need to assign them to workers. The fact that two jobs can help each other is also a known. The problem is formulated in terms of graphs as follows. Suppose $G=(V,E)$ is a graph and $U$ a subset of $V$. A $\beta$-assignment of $G$ with respect to $U$ is an edge set $X$ such that $\deg_x(v)=1$ for all vertices $v$ in $U$. The $\beta$-assignment problem is to find a $\beta$-assignment $X$ such that $\beta(X)=\max\{\deg_x(v):v\in V-U\}$ is minimum. Note that in the case of $G$ as a bipartite graph $(S,T,E)$, we choose $U=S$. In the case of a general graph $G=(V,E)$, an edge $(x,y)$ with $x$ and $y$ in $U$ means the jobs $x$ and $y$ can help each other. Note that we may assume a $\beta$-assignment $X$ contains no edge $(z,w)$ with $z$ and $w$ in $V-U$, since the deletion of such an edge from $X$ never increases the value $\beta(G)$.

Figure 1 shows an example of a graph with 12 vertices, in which the circled vertices form the job set $U=\{1, 2, 3, 4, 7, 9, 10, 11, 12\}$ and the squared vertices form the worker set $V-U=\{5, 6, 8\}$. The bold edges in Fig. 1(b) form a $\beta$-assignment $Y$ of $G$ with respect to $U$. In this case, jobs 1 and 2 help each other and need not be assigned to any worker, job 3 is assigned to worker 6, jobs 4 and 7 help each other, jobs 9 and 10 are assigned to worker 8, and jobs 11 and 12 help each other. Since the maximum number of jobs a worker is assigned is two, we have $\beta(Y)=2$. Note that $Y'=Y-\{(5,6)\}$ is also a $\beta$-assignment with

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Chang and Lee [6] gave an \(O(n^4)\)-time algorithm for the \(\beta\)-assignment problem in bipartite graphs. Chang [7] gave an \(O(n^4)\)-time algorithm for the weighted \(\beta\)-assignment in bipartite graphs. Chang and Ho [8] improved the above results by applying Hungarian-type algorithms for the cardinality, bottleneck and weighted \(\beta\)-assignment problems in bipartite graphs. The running times of these algorithms are all \(O(n^3)\). The main purpose of this paper is to give an \(O(n^3)\)-time algorithm for the \(\beta\)-assignment problem in general graphs. As byproducts, we obtain a duality theorem as well as a necessary and sufficient condition for the existence of a \(\beta\)-assignment for a general graph. The latter result is a generalization of Tutte's theorem [9] for the existence of a perfect matching of a general graph.

2. THE ALGORITHM

This section gives an \(O(n^3)\)-time algorithm for the \(\beta\)-assignment problem in general graphs. This algorithm shares the same significance as the algorithms for the matching problem in general graphs [10–14]. We start this section with some basic definitions.

For a given graph \(G=(V,E)\) and a subset \(U\) of \(V\), a partial \(\beta\)-assignment of \(G\) with respect to \(U\) is an edge subset \(X\) of \(E\) such that \(\deg_x(v)\leq 1\) for all vertices \(v\) in \(U\). For a partial \(\beta\)-assignment \(X\), a vertex \(i\) in \(U\) is exposed if \(\deg_x(i)=0\), a vertex \(j\) in \(V- U\) is safe if \(\deg_x(j)< \beta(X)\), and any other vertex is saturated. We also call \(X\) a partial \(\beta\)-assignment of \(S\), where \(S\) is the set of all nonexposed vertices in \(U\). Figure 1(a) shows a partial \(\beta\)-assignment \(X\) with \(\beta(X)=2\), in which 1 and 7 are exposed, 5 is safe, and all other vertices are saturated.

A walk or \(v_0-v_r\) walk is a sequence \((v_0,v_1,...,v_r)\) of vertices such that \((v_i,v_{i+1})\) is an edge for \(1\leq i\leq r\); where \(v_0\) is called the origin and \(v_r\) the terminus of the walk. A trail (respectively, path) is a walk with no repeated edges (respectively, vertices). Note that a path is always a trail, but the converse is not so. An \(X\)-alternating trail (respectively, path) is a trail (respectively, path) \((v_0,v_1,...,v_r)\) such that one of the sets \((v_{i-1},v_i):1\leq i\leq r\) and \(i\) is odd} and \((v_{i-1},v_i):1\leq i\leq r\) and \(i\) is even} is a subset of \(E- X\) and the other a subset of \(X\). An \(X\)-augmenting trail (respectively, path) is an \(X\)-alternating trail (respectively, path) whose origin \(x\) is an exposed vertex in \(U\) and whose terminus \(y\) is exposed in \(U\) or safe in \(V-U\) when the trail (path) is of odd length, and \(y\) is in \(V-U\) when the trail (path) is of even length. In Fig. 1(a), \(P_1=(1,2,4,7), P_2=(1,2,4,5), P_3=(1,2,4,5,6,9,8)\) are \(X\)-augmenting paths; \(P_4=(1,2,4,8,9,12,11,10,8)\) and \(P_5=(1,2,4,8,9,12,11,10,8,7)\) are \(X\)-augmenting trails. Note that \(P_1\) and \(P_3\) are of odd length with terminus 7 exposed in \(U\), \(P_2\) is of odd length with terminus 5 safe in \(V-U\), and \(P_3\) and \(P_5\) are of even length with terminus 8 in \(V-U\).

Note that in the following contents we sometimes use ‘trail’ and at other times use ‘path’ on purpose. The first reason for using ‘trail’ is that a blossom is an \(X\)-alternating trail. The second reason is that in Lemma 2.2, an \(X\)-augmenting trail may exist when there is a blossom. So, in Lemmas 2.1 and 2.3, we need to deal with \(X\)-augmenting trails. In fact, in this paper, the only \(X\)-augmenting trails that are not paths contain only a vertex that is repeated twice.

An easy argument proves the following lemma. The symmetric difference between two sets \(A\) and \(B\), denoted by \(A\Delta B\), is \((A-B)\cup (B-A)\).
Lemma 2.1. If X is a partial $\beta$-assignment of S and P an X-augmenting trail starting at vertex $i \in U - S$, then $X \Delta P$ is a partial $\beta$-assignment of $S' \supseteq S \cup \{i\}$ and $\beta(X \Delta P) \leq \beta(X)$.

As an example, in Fig. 1, X is a partial $\beta$-assignment and $Y = X \Delta P_1$ is a $\beta$-assignment such that $\beta(Y) = \beta(X) = 2$, where $P_1 = (1, 2, 4, 7)$.

For a partial $\beta$-assignment X of G with respect to U, an X-alternating tree is a rooted tree whose root is an exposed vertex in U and any path from that root to a vertex in the tree is an X-alternating path.

Suppose that X is a partial $\beta$-assignment of G with respect to U. A blossom B of G is an X-alternating trail of odd length that starts and ends at a same vertex b, which is also a unique vertex in B such that the X-alternating trail traverses it twice. We call b the base of the blossom. If there exists an X-alternating path P that starts at an exposed vertex $r \in U$ and terminates at b such that $V(P) \cap V(B) = \{b\}$, and the concatenation of P and B is still an X-alternating trail, then we call P the stem of the blossom and r the root of the blossom. Note that in the case where P contains exactly one vertex, the root r is the same as the base b. The first and the last edges of a blossom B are either both in X or both not in X, since B is of odd length. In the case in which the base b of B is in U, these two edges must both be not in X, otherwise deg$_X(b) \geq 2$ is a contradiction. In Fig. 1(a), (4, 5, 6, 9, 8, 4) is a blossom with base 4, stem (1, 2, 4), and root 1; (8, 9, 12, 11, 10, 8) is a blossom with base 8, stem (1, 2, 4, 8), and root 1; (7, 4, 2, 3, 6, 9, 8, 7) is a blossom with base 7, stem (7), and root 7.

Lemma 2.2. Suppose B is a blossom with root r and base b. If B contains a vertex t in $V - U$, then there exists an X-augmenting trail starting at r.

Proof. Two X-alternating trails always exist in B from b to t of even and odd lengths, respectively; i.e. the clockwise and counterclockwise traversals of B. The concatenation of the stem P and one appropriate X-alternating trail is an X-alternating trail of even length from r to t. By definition, it is an X-augmenting trail starting at r. QED.

Remark. When $t = b$, the X-augmenting trail in the above lemma is in fact a path. The trail is not a path only when $t = b$. In this case, t is the only vertex that appears twice in the trail.

Now we are ready to describe our algorithm for the $\beta$-assignment problem in general graphs. The algorithm begins with an empty partial $\beta$-assignment. Suppose the partial $\beta$-assignment X we have so far is not a $\beta$-assignment. An exposed vertex r in U is taken as the root of an X-alternating tree and vertices and edges are added to the tree by means of a labeling technique. In other words, we say that a vertex i is in the X-alternating tree if it is labeled as 'S:i' or 'T:i'; we also say that i has an S-label or T-label, respectively. We start by labeling the root r as 'S:r'. At each iteration, we scan a labeled but unscanned vertex i and process its neighbors as follows.

Suppose i has an S-label. For any vertex j with $(i,j) \in E - X$, consider the following cases. For the case in which j is unlabeled, we consider two subcases. If j is a nonexposed vertex in U or a saturated vertex in $V - U$, label j as 'T:j'. If j is an exposed vertex in U or a safe vertex in $V - U$, an X-augmenting path has been detected. We can obtain the inverse of this path by backtracking from j using labels, i.e. $j, f(j), f(f(j)), ..., r$. For the case in which j has a T-label, nothing need be done.

Suppose i has a T-label. For any vertex j with $(i,j) \in X$, consider the following cases. For the case in which j is unlabeled and in U, label j as 'S:j'. For the case in which j is unlabeled and in $V - U$, we can find an X-augmenting path from i to j as above. For the case in which j has a T-label, we can form a blossom containing $(i,j)$ as above. For the case in which j has an S-label, we do nothing.

Whenever a blossom B is found, we check whether it contains a vertex t in $V - U$. If this is the case, there exists an X-augmenting trail by Lemma 2.2. On the other hand, suppose all vertices of the blossom are in U. Neither the first nor the last edges of B are in X. Consequently, the base b has an S-label. In this case, we 'shrink' the blossom B, i.e. identify all vertices of B in G as a vertex s to form G/B. In other words, $G/B$ is defined as $(V',E')$ such that $V' = V - V(B) + s$ and $E' = E - \{(u,v) : u \text{ or } v \in V(B)\} + (u, s) : (u,v) \in E$ and $v \in B$, but $u \notin B$. The vertex s in G/B corresponding to B is called a pseudo vertex, and is given the same label as the base of the blossom for the purpose of further tree construction. Note that
blossoms may be shrunken within blossoms to a depth of several levels.

If an augmenting trail is found in the X-alternating tree at some time, there is an X-augmenting trail in the original graph G. The existence of such a trail is proven by repeatedly applying Lemma 2.3.

**Lemma 2.3.** Let B be a blossom with respect to X in G and X' = X - E(B). If there exists an X'-augmenting trail P in G/B, then there exists an X-augmenting trail in G.

**Proof.** If P does not pass through the pseudo vertex s corresponding to B, then P is also an X-augmenting trail in G. If P passes through the pseudo vertex s, then P is of the form P = (u,s)P1(s,v)P2, where P1 and P2 are X-alternating trails. Since s has an S-label in G/B, (u,s) and (s,v) are X-alternating. Since all vertices in the shrunken blossom B are in U, u is adjacent to the base b of B in G, i.e. (u,b) is not in E(G).

Suppose that t in V(B) is such that (t,v) is in E(G). There is always an X-alternating trail P from b to t of even length. Thus, either P1(u,b)P(t,v)P2 or P(t,v)P2 is an X-augmenting trail in G. QED.

Therefore, whenever an X-augmenting trail is found in our labeling procedure, we expand all blossoms within it to obtain an X-augmenting trail P in G. We can then augment X to XAP, and start a new iteration of the algorithm.

When it is impossible to add more labeled vertices or edges into the X-alternating tree L, we call L a Hungarian tree. If any vertex in L \ (V - U) exists, then it is saturated and has a T-label. In this case we backtrack to obtain an X-alternating path of odd length from i to r. As in the case of an X-augmenting path, we expand the blossoms, if necessary, to create an X-alternating trail of odd length from i to r in the original graph G. Because we have no other better choice, we replace X with XAP and thus the value of \( \beta(X) \) is increased by one. In the case of L \ (V - U) = \emptyset, we will prove that G has no \( \beta \)-assignment with respect to U. The tree-building procedure is repeated for at most |U| iterations until an optimum \( \beta \)-assignment with respect to U has been obtained or at some iteration we may conclude that G has no \( \beta \)-assignment with respect to U. More precisely, we have the following algorithm. An example is illustrated in Figs 2–4.

**Algorithm Beta-Assignment** (G,U)

1. \( X \leftarrow \phi; k \leftarrow 0; \)
2. for each exposed vertex \( r \in U \) do
3. \( \text{label } r \text{ with } 'S'; \)
4. \( (\ast) \text{ find a labeled but unscanned vertex } i \in V \text{ and scan it as follows; } \)
5. if no such vertex i then {
6. case 1. there is a labeled vertex \( j \in V - U: \)
7. backtracking \( j \) to \( r \) using labels to get an X-alternating path \( P; \)
8. expand blossoms in it;
9. \( X \leftarrow XAP; k \leftarrow k + 1; \text{ goto Next-Iteration; } \)
10. case 2. all labeled vertices are in \( U \) or are pseudo vertices:
11. \( G \) has no \( \beta \)-assignment with respect to \( U; \) \text{ halt};
12. if i has an S-label then {for each \((i,j) \notin X \) do
13. \( \text{A Hungarian Tree } \)
14. Initially \( X_0 = \emptyset \) and \( \beta(X_0) = 0. \)
15. There is an \( X_1 \)-augmenting path \( P_0 = (12,13). \)
16. Augment \( X_0 \) using \( P_0 \) to get \( X_1 \) with \( \beta(X_1) = 1. \)

Fig. 2. A graph of 18 vertices with \{7, 13\} being the set of workers.
The $\beta$-assignment problem

$X_8$ has 8 edges and $\beta(X_8) = 2$. Since $B_2 \cap (V - U) = 7$, there exists an $X_8$-augmenting path $P_8 = (11, 2, 3, 4, 5, 6, 7)$. Augment $X_8$ by $P_8$ to get $X_9$ with $\beta(X_9) = 2$.

Fig. 3. After eight iterations.

case 1. $j$ has a $T$-label: do nothing;

case 2. $j$ has an $S$-label: goto Blossom-Subroutine;

case 3. $(j \in U$ and $j$ is exposed) or $(j \in V - U$ and $j$ is safe): goto Augmenting-Subroutine;

case 4. $j$ is saturated or nonexposed: label $j$ as ‘$T_1$’; endfor};

if $i$ has a $T$-label then {for each $(ij) \in X$ do

case 1. $j$ has an $S$-label: do nothing;

case 2. $j$ has a $T$-label: goto Blossom-Subroutine;

case 3. $j \in V - U$: goto Augmenting-Subroutine;

case 4. $j \in U$: label $j$ by ‘$S_1$’; endfor};

goto (*);

Next-Iteration: erase all vertex labels;

endfor;

output $(X, k)$;

Blossom-Subroutine

[backtrack from $i$ and $j$ using labels until $r$ is reached to find a blossom $B$;

if $B$ contains a vertex $j \in V - U$

then goto Augmenting-Subroutine;

else (shrink $B$ to a pseudo vertex; give the pseudo vertex the label of the base of $B$; leave it unscanned; goto (*));

After backtraking from 13, we find an $X_9$-augmenting path $P_9 = (13, 14, 18, 17, 16, 15, 7, 9, 10, 11, 2, 1)$.

Augment $X_9$ by $P_9$ to get $X_{10}$.

Stop since there are no exposed vertices.

Fig. 4. The final iteration.
Augmenting-Subroutine
(backtrack from \( j \) using labels to \( r \) in order to find an \( X \)-augmenting path \( P \); expand blossoms in it; \( X \leftarrow X \Delta P \); goto Next-Iteration;)
end Beta-Assignment.

3. CORRECTNESS OF THE ALGORITHM

We will prove the correctness of the algorithm with a primal-dual approach. As by-products of this proof, we also obtain a strong duality theorem as well as a necessary and sufficient condition for the existence of a \( \beta \)-assignment for a general graph. Since a perfect matching in \( G=(V, E) \) is precisely a \( \beta \)-assignment with respect to \( V \), what we find is a generalization of Tutte's theorem for perfect matchings.

An odd family (OF for short) is a family \( \text{OF}_n=\{V_1,V_2,...,V_m\} \) of disjoint sets of vertices of \( U \), such that each \( |V_i| \) is odd and no vertex in \( V_i \) is adjacent to a vertex in \( V_j \) when \( i \neq j \). Figure 5 shows an example of an odd family. Denote as \( \eta_m \) the set of vertices \( \bigcup_{i=1}^{m} V_i \). For any \( W \subseteq \text{V} \), the neighborhood \( N(W) \) of \( W \) is \( \{x \in \text{V} - W: x \text{ is adjacent to some } y \in W\} \). The following lemma is a consequence of the definition of the odd family.

**Lemma 3.1.** For any \( \beta \)-assignment \( X \), there are at least \( m \) vertices in \( \eta_m \) that can only be assigned through \( X \) to vertices in \( N(\eta_m) \).

Let \( X \) be a \( \beta \)-assignment of \( G \) with respect to \( U \) and let \( \text{OF}_n=\{V_1,V_2,...,V_m\} \) be an odd family. By Lemma 3.1, there are at least \( m \) vertices of \( \eta_m \), which can only be assigned through \( X \) to vertices in \( N(\eta_m) \). Since \( \eta_m \cap N(\eta_m) = \emptyset \), there are at least \( m = |N(\eta_m) \cap U| \) vertices of \( \eta_m \), which must be assigned to vertices of \( N(\eta_m) \cap U \), where \( \bar{U} = \text{V} - U \). This implies that

\[
\beta(X) = \max_{\nu \in \bar{U}} \deg_x(\nu) \geq \max_{\nu \in N(\eta_m) \cap \bar{U}} \deg_x(\nu) \geq \frac{m - |N(\eta_m) \cap U|}{|N(\eta_m) \cap U|}.
\]

So we have the following min–max duality inequality.

**Lemma 3.2.**

\[
\min_{X \beta - \text{G}(U)} \beta(X) \geq \max_{\eta_m \neq \emptyset} \frac{m - |N(\eta_m) \cap U|}{|N(\eta_m) \cap U|},
\]

where \( \beta - \text{G}(U) \) is a abbreviation of \( \beta \)-assignment of \( G \) with respect to \( U \).

Note that Lemma 3.2 and the odd family in Fig. 5 permit us to conclude that the \( \beta \)-assignment \( X_{10} \) in Fig. 4(b) is an optimum \( \beta \)-assignment for the graph \( G \) in Figs 2–4. Denote as \( \text{OF}_s \) the odd family in Fig. 5. Then, \( \eta_s = \{1,8,9,11,12,14,15,16,17,18,19\} \), \( N(\eta_s) \cap U = \{(2,10)\} \), and \( N(\eta_s) \cap \bar{U} = \{7,13\} \). According to Lemma 3.2,
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\[
3 = \beta(x_{10}) \geq \beta(G) \geq \max_{u,\eta \in \mathfrak{p}} \left| \frac{m - |\mathcal{N}(\eta_{m}) \cap U|}{|\mathcal{N}(\eta_{m}) \cap \overline{U}|} \right| \geq \frac{7 - |\mathcal{N}(\eta_{7}) \cap U|}{|\mathcal{N}(\eta_{7}) \cap \overline{U}|} = 3.
\]

Therefore, the inequalities are actually equalities. This proves that $X_{10}$ is an optimum $\beta$-assignment of $G$ with respect to $U$ and $\beta(G) = 3$. We shall use the same idea to prove that the output of Algorithm Beta-Assignment is an optimum $\beta$-assignment.

Suppose $L$ is a Hungarian tree at some iteration, and $S$ and $T$ are the sets of all vertices in the shrunken graph $G^*$ with $S$-labels and $T$-labels, respectively. We require the following lemma to show the correctness of our algorithm and that the inequality in Lemma 3.2 is in fact an equality.

**Lemma 3.3.** The following statements are true.

1. All vertices in a shrunken blossom are contained in $U$.
2. Each vertex with an $S$-label in $G^*$ is either in $U$ or is a pseudo vertex. A pseudo vertex always has an $S$-label.
3. Every vertex, except the root $r$, with an $S$-label in $G^*$ is not exposed.
4. Every vertex with a $T$-label in $G^*$ is saturated or nonexposed.
5. $N_{G^*}(S \cap L) = T \cap L$.
6. If $S \cap L = \{v_1, v_2, \ldots, v_m\}$, then $O_m = \{V_1, V_2, \ldots, V_m\}$ is an odd family, where
   
   \[
   V_i = \begin{cases} \{v_i\}, & \text{if } v_i \text{ is a vertex in } G, \\ \text{the set of vertices in the corresponding blossom}, & \text{if } v_i \text{ is a pseudo vertex}. \end{cases}
   \]

1. $|S \cap U| = |T \cap L \cap U| + klT \cap L \cap \overline{U}| + 1$, where $k = \beta(X)$.

**Proof:**

1. According to the blossom subroutine, the algorithm does not shrink a blossom when it contains a vertex not in $U$.
2. Any nonpseudo vertex is labeled as "$S:*" only at lines 1 or 4 of the algorithm. In any case, it is in $U$. On the other hand, suppose $s$ is a pseudo vertex whose corresponding blossom is $B$. According to step (1), all vertices in $B$ are in $U$. So neither the first nor the last edges of $B$ are in $X$, otherwise $\deg_x(b) \geq 2$ for the base $b$ of $B$. Consequently, $b$ has an $S$-label and so does $s$.
3. The root $r$ of $L$ is clearly exposed in the algorithm. Suppose some vertex $i$, other than the root, has an $S$-label. For the case in which $i$ is not a pseudo vertex, $i$ is labeled as "$S:*" only at line 4. This happens only when there is some $j \in U$ with $(i, j) \in X$. Hence $i$ is nonexposed. For the case in which $i$ is a pseudo vertex, the base of its corresponding blossom has an $S$-label according to step (2). According to the above argument, $b$ is nonexposed and so is $i$.
4. A vertex $j$ is labeled as "$T:*" only at line 3 of the algorithm, so $j$ is nonexposed (respectively saturated) when it is in $U$ (respectively $V - U$).
5. In accordance with the algorithm, each vertex $j$ adjacent to a vertex $i$ of $S$-label via an edge $(i, j) \in X$ is either in $T \cap L$ or in $S \cap L$; and in the latter case, $i$ and $j$ detect a blossom. Then according to steps (1)–(3), every vertex $j$ with an $S$-label is in $U$ or is a pseudo vertex, and there is exactly one edge $(i, j)$ in $X$, except when $j$ is the root $r$. Using line 4 of the algorithm, $i$ has a $T$-label. Hence $N_{G^*}(S \cap L) \subseteq T \cap L$. Conversely, using line 3 of the algorithm, $T \cap L \subseteq N_{G^*}(S \cap L)$. So, $N_{G^*}(S \cap L) = T \cap L$.
6. Since the number of vertices in a blossom is odd, each $|V_i|$ is odd when we expand any blossom in a blossom or a pseudo vertex. Suppose $i \neq j$ such that $V_i$ has a vertex adjacent to some vertex in $V_j$. Then, $v_i$ is adjacent to $v_j$ in the shrunken graph. As in the second line of the proof of step (5), $(v_i, v_j)$ would detect a blossom and so must be shrunk into a vertex, which is a contradiction. So $O_m = \{V_1, V_2, \ldots, V_m\}$ is an odd family.
7. Using step (5), $T \cap L = N(S \cap L)$. Using steps (2) and (3), each vertex $j$ in $S \cap L$, except the root, is adjacent to exactly one vertex in $T \cap L$. With step (4), each vertex in $T \cap L \cap U$ (respectively $T \cap L \cap \overline{U}$) is adjacent to exactly one (respectively $k$) vertex in $S \cap T$. So, $|S \cap U| = |T \cap L \cap U| + klT \cap L \cap \overline{U}| + 1$. QED.

**Lemma 3.4.** The following statements are equivalent.
1. $G$ has a $\beta$-assignment with respect to $U$.
2. Algorithm Beta-Assignment does not halt at line 2.
3. Algorithm Beta-Assignment outputs a $\beta$-assignment $X$ of $G$ with respect to $U$.

Proof. (1)$\Rightarrow$(2). Suppose the algorithm halts at line 2. Let $L^*$ be the Hungarian tree at that moment, i.e. all vertices in $L^*$ are in $U$ or are pseudo vertices. According to Lemma 3.3, steps (5)-(7), $O^*_m$ is an odd family, $IS \cap L^* = IT \cap L^* + 1$, and $N_G(S \cap L^*) = T \cap L^*$. Using Lemma 3.1, in $\eta_m$, there are at least $m$ vertices that can only be assigned through $X^*$ to vertices in $N(r^*/U) = NG$. Thus, $IS(\eta_m) \subseteq T \cap L^*$. Yet $IT \cap L^* < IS \cap L^*$, at least two vertices of $\eta_m$ must be assigned to the same vertex in $T \cap L^* \subseteq U$, which is impossible. Thus, $G$ has no $\beta$-assignment with respect to $U$.

(2)$\Rightarrow$(3). It is clear that, after at most $\lvert U \rvert$ iterations, a $\beta$-assignment $X$ of $G$ with respect to $U$ together with $\bar{X}$ is obtained by the algorithm.

(3)$\Rightarrow$(1). This is clear. QED.

Now, we are ready for simultaneous proof of the strong duality theorem and the validity of the algorithm.

**Theorem 3.5.** Algorithm Beta-Assignment works and the strong duality equality holds:

$$\min_{x,B(G,U)} \beta(X) = \max_{\omega, \mu} \frac{m - \lvert N(\eta_m) \cap U \rvert}{\lvert N(\eta_m) \cap U \rvert}.$$

Proof. Suppose $\min_{x,B(G,U)} \beta(X) = \infty$, i.e. $G$ has no $\beta$-assignment with respect to $U$. Lemma 3.4 shows that the algorithm does halt without returning any $\beta$-assignment. Also, as in the proof of Lemma 3.4, there exists a Hungarian tree $L^*$ whose vertices are in $U$ or are pseudo vertices such that $S \cap L^*$ produces an odd family $O^*_m$ with $m^* = IS \cap L^* = IT \cap L^* + 1$ and $N_G(\eta_m) = N_G(S \cap L^*) = T \cap L^* \subseteq U$. Thus,

$$\frac{m^* - \lvert N(\eta_m) \cap U \rvert}{\lvert N(\eta_m) \cap U \rvert} = 1 = \infty.$$

Therefore, the equality holds when $G$ has no $\beta$-assignment with respect to $U$.

On the other hand, suppose $G$ has a $\beta$-assignment. Let $L^*$ be a Hungarian tree that forces the $k$ value to increase from $k^* - 1$ to $k^*$. Let $O^*_m$ be the odd family produced by $S \cap L^*$, as in Lemma 3.3, step (6). According to Lemma 3.3, step (5), $N_G(S \cap L^*) = T \cap L^*$ and so $N_G(\eta_m) = T \cap L^*$. Using Lemma 3.3, step (7), $m^* = IS \cap L^* = IT \cap L^* \cap U + (k^* - 1)IT \cap L^* \cap U + 1$. Hence,

$$\frac{m^* - \lvert N(\eta_m) \cap U \rvert}{\lvert N(\eta_m) \cap U \rvert} = k^*.$$

Moreover,

$$k^* = \beta(X^*) = \min_{x,B(G,U)} \beta(X) = \max_{\omega, \mu} \frac{m - \lvert N(\eta_m) \cap U \rvert}{\lvert N(\eta_m) \cap U \rvert} \geq \frac{m^* - \lvert N(\eta_m) \cap U \rvert}{\lvert N(\eta_m) \cap U \rvert}.$$

Therefore, the inequalities are equalities. This proves that $X^*$ is an optimum solution and the strong duality equality holds. QED.

By a similar argument, we have the following existence theorem for a $\beta$-assignment of a general graph.

**Theorem 3.6.** $G = (V,E)$ has a $\beta$-assignment with respect to $U$ if and only if $\theta(G-D) \leq \lvert D \rvert$ for all $D \subseteq U$, where $\theta(G-D)$ is the number of odd components contained in $U$ of $G - D$.

Proof. Suppose there exists a subset $D \subseteq U$ such that $\theta(G-D) > \lvert D \rvert$. Consider the odd family $O_m = \{V_1, V_2, ..., V_m\}$ in which each $V_i$ is an odd component contained in $U$ of $G - D$. Then, $N(\eta_m) \subseteq D$ and...
so \( m = (G - D)|D| \geq |\{ \gamma \}|. \) Thus, \( G \) has no \( \beta \)-assignment with respect to \( U \) according to Lemma 3.1.

Conversely, suppose \( G \) has no \( \beta \)-assignment with respect to \( U \). According to Lemmas 3.3 and 3.4, there exists a Hungarian tree \( L^* \) whose vertices are in \( U \) or are pseudo vertices such that \( S \cap L^* \) produces an odd family \( O^*_x \) with \( m^* = |S \cap L^*| = |T \cap L^*| + 1 \) and \( N_0(\eta_x) = N_0(S \cap L^*) = T \cap L^* \subseteq U \). Since \( V \subseteq U \) for each \( i \), \( \theta(G - (T \cup L^*)) \geq m^* = |L^* \cap \Omega| + 1 > |L^* \cap \Omega| \). QED.

As mentioned before, a perfect matching in \( G = (V, E) \) is just a \( \beta \)-assignment with respect to \( V \). Thus, Theorem 3.6 is a generalization of Tutte's theorem for perfect matchings.

4. IMPLEMENTATION AND TIME COMPLEXITY OF THE ALGORITHM.

There are two principal elaborations required for Algorithm \textbf{Beta-Assignment}. First, it must detect and shrink blossoms. Second, it must be able to discover appropriate \( X \)-augmenting trails through shrunken blossoms. Edmonds [11] gave efficient implementations of these two operations for the matching algorithm. Although a matching is only a special \( \beta \)-assignment with respect to \( U \) when \( U = V \), Edmonds' implementations of those two operations are still applicable to our algorithm since our algorithm shrinks blossoms that contain only vertices in \( U \). According to Lawler's modification [4] of Edmonds' blossom procedure, the time complexity of those operations are listed as follows.

Our algorithm requires the following blossom operations: (1) blossom detecting; (2) blossom backtracking; (3) blossom checking (to see if the blossom is contained in \( U \)); (4) extra labeling of vertices in blossoms (applying missing labels to them); (5) blossom recording. The first operation needs \( O(1) \) time and the other operations can be done in \( O(n) \) time.

Our algorithm also requires the following augmenting trail operations: (1) \( X \)-augmenting trail detecting; (2) backtracking; (3) finding appropriate \( X \)-alternating trails through shrunken blossoms; (4) augmenting the partial \( \beta \)-assignment \( X \). All of these operations require \( O(n) \) time, except for the first operation, which only needs constant time.

We can now consider the time complexity of our algorithm. In each tree-building iteration, we apply the labeling procedure to \( G \). No more than \( O(n) \) blossoms in \( U \) are formed. Each blossom requires the shrinking operation, which needs \( O(n) \) time. There is at most one \( X \)-augmenting trail formed in each iteration. It requires \( O(n) \) time to backtrack the trail and augment \( X \). Moreover, each application of the labeling procedure scans at most \( O(n) \) labeled but not scanned vertices, and each scanning operation requires at most \( O(n) \) steps. Hence, simple scanning and labeling operations contribute \( O(n^2) \) steps per iteration. Since there are no more than \( O(|U|) \) iterations, we can conclude that the overall running time of Algorithm \textbf{Beta-Assignment} is \( O(n^2) \).

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