Topological properties of twisted cube 1

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Abstract

Twisted cube, $TQ_n$, is derived by changing some connections of hypercube $Q_n$ according to specific rules. Recently, many topological properties of this variation cube are studied. In this paper, we prove that its connectivity is $n$, its wide diameter and fault diameter are $\lceil n/2 \rceil + 2$. Furthermore, we show that $TQ_n$ is a pancyclic network that is cycles of an arbitrary length at least four. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

Network topology is a crucial factor for interconnection network since it determines the performance of the network. Many interconnection network topologies have been proposed in the literature for the purpose of connecting hundreds or thousands of processing elements [3,4,6,10]. Network topology is always represented by a graph where nodes represent processors and edges represent links between processors. Among these topologies, the binary hypercube, $Q_n$, is one of the most popular topology. However, $Q_n$ does not make the best use of its hardware in the following sense: given $N = 2^n$ nodes and $nN/2$ links, it is possible to fashion networks with lower diameters than the

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hypercube's diameter \(n\). One of such topologies is called twisted cube \([7]\), \(TQ_n\), which is derived by changing the connection of some links of the hypercube according to some specified rules. The diameter of twisted cube topology is \([(n+1)/2]\), almost a factor of 2 improvement. This is achieved by forfeiting some of the hypercube's high degree of symmetry and redundancy. Recently, many topological properties of this variation cube are studied in the literature \([1,2]\).

In order to evaluate the performance of a network topology, we can consider the following measures: vertex connectivity, diameter, wide diameter, fault diameter, and embedding of cycles. The vertex connectivity (simply abbreviated as connectivity) of a network \(G = (V, E)\), denoted by \(\kappa(G)\) or \(\kappa\), is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's theorem that there always exist \(\kappa\) internally vertex-disjoint (abbreviated as disjoint) paths between any two vertices. Disjoint paths between a pair of vertices contribute to multipath communication between these two vertices and provide alternative routes in the case of node or link failures. Thus large connectivity is preferred.

Wide diameter and fault diameter were proposed in \([5,8]\). For any pair of vertices, say \(u\) and \(v\), we find \(\kappa\) disjoint paths such that the longest path length of \(\kappa\) disjoint paths is minimum, denoted by \(d_\kappa(u, v)\), among all possible choices of \(\kappa\) disjoint paths. The wide diameter is defined as the maximum of \(d_\kappa(u, v)\) over all \(u, v \in V\). Small wide diameter is preferred since it enables fast multipath communication. Fault diameter estimates the impact on diameter when faults occur, i.e., removal of vertices from \(G\). For a pair of vertices \(u\) and \(v\), we find the maximum of shortest path length between \(u\) and \(v\) over all possible \(\kappa - 1\) faults, denoted by \(d_{\kappa-1}^f(u, v)\). The \((\kappa - 1)\)-fault diameter is the maximum of \(d_{\kappa-1}^f(u, v)\) for all \(u, v \in V\), i.e., the maximum transmission delay of \(\kappa - 1\) faults. Small \((\kappa - 1)\)-fault diameter is also desirable to obtain smaller communication delay when faults occur. Wide diameter and fault diameter of a twisted cube \(TQ_n\) are studied in this paper.

An important aspect of \(TQ_n\) is its ability of efficiently simulating computations on other networks, which is portability of algorithms from other parallel interconnection structures, such as cycle or tree, to \(TQ_n\). Such simulations can be reduced to graph embedding problem. We also consider the problem of embedding cycles architectures in twisted cubes.

Most of the graph definitions used in this paper are standard (see \([9]\)). Let \(G = (V, E)\) be a finite, undirected graph. Throughout this paper, node and vertex are used interchangeably to represent the element of \(V\). Edge and link are used interchangeably to represent the element of \(E\). Let \(C_i\) denote the length of a cycle. The distance between vertices \(u\) and \(v\), denoted by \(d_G(u, v)\), is the length of the shortest path from \(u\) and \(v\).

The rest of this papers is organized as follows. In Section 2 we discuss some basic topological properties of twisted cubes. The connectivity, fault diameter and wide diameter for twisted cubes of odd dimension are studied in Section 3.
Embedding of cycles into twisted cubes is presented in Section 4. Finally, we make concluding remarks in Section 5.

2. Twisted cube topology and its properties

The \( n \)-dimensional hypercube, \( Q_n \), consists of all the binary \( n \)-bit strings as its vertex set and two vertices \( u \) and \( v \) are adjacent if and only if \( u \) differs from \( v \) by exactly one bit. Let \( u = u_{n-1}u_{n-2} \ldots u_0 \) and \( v = v_{n-1}v_{n-2} \ldots v_0 \) be two vertices of \( Q_n \). \((u,v)\) is an edge in \( E(Q_n) \) of dimension \( i \) if the \( i \)th bit of \( u \) is different from that of \( v \). Twisted cube was first defined by Hilbers et al. [7]. A twisted \( n \)-cube, denoted by \( TQ_n \), is a variant of \( n \)-dimensional hypercube \( Q_n \). \( TQ_n \) has the same number of nodes and edges as \( Q_n \). We restrict the following discussion on \( TQ_n \) for the case that \( n \) is odd. Let \( n = 2m + 1 \), to form the twisted cube, we remove some links from the hypercube and replace them with links that span two dimensions in such a manner that the total number of links \((nN/2)\) is conserved.

To be precise, let \( u = u_{n-1}u_{n-2} \ldots u_0 \) be any vertex in \( TQ_n \). We define the parity function \( P_2(u) = u_i \oplus u_{i-1} \oplus \cdots \oplus u_0 \), where \( \oplus \) is the exclusive-or operation. If \( P_{2j-2}(u) = 0 \) for some \( 1 \leq j \leq m \), we divert the edge on \((2j-1)\)th dimension to node \( v \) such that \( v_{2j-1} = u_{2j-1} \) and \( v_i = u_i \) for \( i \neq 2j \) or \( 2j - 1 \). Such diverted edges is called twisted edges. \( TQ_3 \) and \( TQ_5 \) are shown in Fig. 1(a) and (b).

We may formally define the term of twisted cube recursively as follows: A twisted 1-cube, \( TQ_1 \), is a complete graph with two vertices, 0 and 1. Let \( n \) be an odd integer and \( n \geq 3 \). We decompose vertices of \( TQ_n \) into four sets \( S^{0,0}, S^{0,1}, S^{1,0} \) and \( S^{1,1} \) where \( S^{ij} \) consists of those vertices \( u \) with \( u_{n-1} = i \) and \( u_{n-2} = j \). For each \((i,j) \in \{(0,0), (0,1), (1,0), (1,1)\}\), the induced subgraph of \( S^{ij} \) in \( TQ_n \) is isomorphic to \( TQ_{n-2} \). Edges which connect these four subtwisted cubes can be described as follows: Any node \( u_{n-1}u_{n-2} \ldots u_1u_0 \) with \( P_{n-3}(u) = 0 \) is connected to \( \bar{u}_{n-1}\bar{u}_{n-2}u_{n-3} \ldots u_0 \) and \( \bar{u}_{n-1}u_{n-2}u_{n-3} \ldots u_0 \); and \( u_{n-1}u_{n-2}u_{n-3} \ldots u_0 \) and \( \bar{u}_{n-1}\bar{u}_{n-2}u_{n-3} \ldots u_0 \), if \( P_{n-3}(u) = 1 \).

The following lemma can be easily obtained from the definition of twisted cubes.

Lemma 1. Let \( u = u_{n-1}u_{n-2} \ldots u_1u_0 \) and \( v = v_{n-1}v_{n-2} \ldots v_1v_0 \) be two vertices of \( TQ_n \) with \((u,v) \in E(TQ_n)\). If \( u_{n-1} = v_{n-1}, u_{n-2} = v_{n-2}, \) and \( P_{n-3}(u) = P_{n-3}(v) \), then \( P_{n-5}(u) = P_{n-5}(v) \).

To discuss the wide diameter and the fault diameter of the twisted cube, we need to review the shortest path routing algorithm [1]. Defining the 0th "double bit" of node address \( u \) to be the single bit \( u_0 \), and the \( j \)th "double bit" to be \( u_{2j}u_{2j-1} \). Let \( u, v \) be any two vertices of \( TQ_n \). We defined the double Hamming distance of \( u \) and \( v \), denoted by \( d_h(u, v) \), to be the number of different double bits between \( u \) and \( v \). Obviously, \( d_{TQ_n}(u,v) \geq d_h(u,v) \).
We can find the shortest path between any two vertices using the algorithm proposed in [1]. Let \( u \) and \( v \) be two vertices of \( TQ_n \). Let \( z = u \). The basic strategy of the algorithm is to recursively find a neighborhood \( w \) of \( z \) that reduces \( h_d(w, v) \). To be precise, the strategy is described as follows.
1. If \( z = v \), then the path is determined.
2. Assume that there exist neighbors \( w \) of \( z \) such that \( h_d(w, v) = h_d(z, v) - 1 \). Let \( w' \) be the such \( w \) that differs from \( z \) with the largest double bit. Then reset \( z \) to be \( w' \).
3. Assume that all the neighborhood \( w \) of \( z \) satisfy \( h_d(w, v) \geq h_d(z, v) \). Let \( j \) be the smallest index of double bits that \( z \) differs from \( v \). Choose \( w' \) to be the neighbor of \( z \) that differs from \( z \) in the \( 2j \)th bit. Then reset \( z \) to be \( w' \).

Since the rightmost differing double bit is selected in step 3, the resulting parity change guarantees that all subsequent routing for the message will be by step 2 until the destination is reached. Hence, step 3 is executed at most once for a given message. With this routing algorithm, we have the following theorems.

**Theorem 1** [7]. The diameter of the twisted cube \( TQ_n \) is \( \lceil (n + 1)/2 \rceil \).

**Theorem 2.** \( h_d(u, v) \leq d_{TQ_n}(u, v) \leq h_d(u, v) + 1 \) for any \( u, v \in V(TQ_n) \).

**Lemma 2.** Let \( u \) and \( v \) be any two different nodes in the same \( S^{i,j} \) of \( TQ_n \) and \( L \) be any shortest path joining \( u \) to \( v \). If \( P_{n-3}(z) = 0 \) for all nodes \( z \) in \( L \), then the length of \( L \) is at most \( \lceil (n - 2)/2 \rceil - 1 \).

**Proof.** Write \( L \) as \( u = u^0, u^1, \ldots, u^k = v \). Suppose that there exists some index \( i \) with \( 0 \leq i \leq k - 1 \) such that \( u^i \) differs from \( u^{i+1} \) in exactly one bit, say \( t \), with \( 0 \leq t \leq n - 3 \). Then \( P_{n-3}(u^i) \neq P_{n-3}(u^{i+1}) \). This is contradiction to the assumption, i.e., \( P_{n-3}(u^i) = 0 = P_{n-3}(u^{i+1}) \). Hence each \((u^i, u^{i+1})\) is either a twisted edge or \( u^i \) differs from \( u^{i+1} \) in the \( \lceil (n - 2)/2 \rceil \)th double bit. Since both \( u \) and \( v \) are in the same \( S^{i,j} \), the length of \( L \) is at most \( \lceil (n - 2)/2 \rceil \) and \( u_0u_1u_2 = v_0v_1v_2 \). Suppose that the length of \( L \) is \( \lceil (n - 2)/2 \rceil \). Since each \((u^i, u^{i+1})\) is a twisted edge, we have \( u_0 = v_0 = 0 \) and \( u_{2j} = v_{2j} \) for \( 1 \leq j \leq \lfloor n/2 \rfloor - 2 \). Based on the definition of double Hamming distance, we have \( h_d(u, v) = \lceil (n - 2)/2 \rceil - 1 \). Applying the shortest path routing algorithm, we can conclude that \( d_{TQ_n}(u, v) = \lceil (n - 2)/2 \rceil - 1 \). We get a contradiction. Hence the length of \( L \) is at most \( \lceil (n - 2)/2 \rceil - 1 \) and the lemma is proved. 

3. **Fault diameter, wide diameter, and connectivity**

We here formally define wide diameter and fault diameter of an underlying network \( G = (V, E) \). For a vertex \( u \) in \( G \), the neighborhood of \( u \), denoted by \( N(u) \), is defined as \( \{ v \mid (u, v) \in E \} \). Let \( u \) and \( v \) be two distinct vertices in \( G \), and let \( \kappa(G) = \kappa \). Let \( C(u, v) \) denote the set of all \( x \) disjoint paths between \( u \) and \( v \). Each element \( i \) of \( C(u, v) \) consists of \( x \) disjoint paths, and the longest length among these \( x \) paths is denoted by \( l_i(u, v) \). The number of elements in
\[ C(u, v) \text{ is denoted by } |C(u, v)|. \]

We define \( d_{\alpha}(u, v) \) as the minimum over all \( l_i \), i.e., \( d_{\alpha}(u, v) = \min_{i \in C(u, v)} l_i(u, v) \). We write \( d_1(u, v) \) as \( d(u, v) \), which means the shortest distance between \( u \) and \( v \). \( D_\alpha(G) \) is called the \( \alpha \)-diameter of \( G \) and is given by

\[ D_\alpha(G) = \max_{u, v \in V} \{ d_\alpha(u, v) \} . \]

By definition, \( D_\alpha(G) = \infty \) if \( \alpha \geq \kappa + 1 \). We usually write \( D_1(G) \) as \( D(G) \) and call \( D(G) \) simply the diameter of \( G \). We are particularly interested in \( D_\kappa(G) \). For a positive integer \( \beta \), \( d_{\beta}^\kappa(u, v) \) is defined as

\[ d_{\beta}^\kappa(u, v) = \max_{\{F \subseteq V : |F| = \beta\}} \{ d(u, v) \text{ in } G - F \mid u, v \notin F \} . \]

The \( \beta \)-fault diameter, denoted by \( D_{\beta}^\kappa(G) \), is given by

\[ D_{\beta}^\kappa(G) = \max_{u, v \in V} \{ d_{\beta}^\kappa(u, v) \} . \]

If \( \beta \geq \kappa \), \( D_{\beta}^\kappa(G) = \infty \) by definition. We are in particular interested in \( D_{\kappa-1}^\kappa(G) \). Obviously, we have \( D(G) \leq D_{\kappa-1}^\kappa(G) \leq D_\kappa(G) \).

It is known that \( \kappa(Q_n) = n \) and \( D_\kappa(Q_n) = D_{n-1}^\kappa(Q_n) = n + 1 \). In this section, we will prove that \( D_n(TQ_n) = D_{n-1}^\kappa(TQ_n) = \lfloor n/2 \rfloor + 2 \) for all odd \( n \). With this result, we can conclude that the connectivity of \( TQ_n \) is \( n \). A node \( u \) of \( TQ_n \), denoted by \( u = 0^i1^{n-1} \), is a binary string of length \( n \) with the first \( i \) 0's and the last \( n-i \) 1's. We first prove the following lemma.

\textbf{Lemma 3.} \( D_{n-1}^\kappa(TQ_n) \geq \lfloor n/2 \rfloor + 2 \), where \( n \) is an odd integer.

\textbf{Proof.} Let \( u = 0^{n-1}, v = 0^21^{n-3}0 \), and \( u' = 010^{n-3}1 \). Assume that the faulty set \( F = N(u) - \{u'\} \). Hence \( |F| = n - 1 \). Obviously, any path that joins \( u \) to \( v \) without traversing any node in \( F \) is a path from \( u \) through \( u' \), then through a neighborhood of \( d(u,v) \), say \( u'' \), and then followed by a path joining \( u'' \) to \( v \). These \( u'' \) are in the set \( W = \{010^{n-2}, 120^{n-3}1\} \cup \{010^{n-4}j101: 0 \leq j \leq n-4\} \). Obviously, \( h_{\alpha}(x, v) = \lfloor n/2 \rfloor \) for any \( x \in W - \{010^{n-2}\} \). By the shortest path algorithm, we can check that \( d_{n-1}^\kappa(y, v) = \lfloor n/2 \rfloor \) where \( y = 010^{n-2} \). Hence the distance between any vertex in \( W \) to \( v \) in \( TQ_n \) is exactly \( \lfloor n/2 \rfloor \). Therefore \( D_{n-1}^\kappa(TQ_n) \geq \lfloor n/2 \rfloor + 2 \). Hence the lemma is proved. \( \square \)

A path \( P: u = u^{(0)}u^{(1)} \ldots u^{(k-1)}u^k = v \) with \( k \geq 3 \) is called a twisted path if \( P_{n-3}(u') = P_{n-3}(u') \) and \( P_{n-1}(u') = P_{n-1}(u') \) for \( 1 \leq i, j \leq k - 1 \). For any node \( u = u_{n-1}u_{n-2} \ldots u_1u_0 \) in \( TQ_n \) and any \((i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\} \), \( u^{(i,j)} \) denotes the node \( iju_{n-3} \ldots u_1u_0 \).

\textbf{Lemma 4.} For any two different vertices \( u \) and \( v \) in \( TQ_n \), there are \( n \) disjoint paths, \( L_1, L_2, \ldots, L_n \), joining \( u \) to \( v \) such that (1) the length of each \( L_i \) is at most \( \lfloor n/2 \rfloor + 2 \), (2) \( L_1 \) is the shortest path joining \( u \) to \( v \), and (3) the length of \( L_i \) is at most \( \lfloor n/2 \rfloor + 1 \) if \( L_i \) is a twisted path.
Proof. The proof is by induction. Obviously, the lemma is true for $n = 1$. For $n \geq 3$, assume that such $k$ disjoint paths exist for any two distinct nodes in $TQ_k$ and any odd $k < n$. Now, we consider any two nodes $u = u_{n-1} u_{n-2} \ldots u_1 u_0$ and $v = v_{n-1} v_{n-2} \ldots v_1 v_0$ in $TQ_n$. We discuss the following six cases.

In cases 1 and 2, both $u$ and $v$ satisfy $v_{n-3} \ldots v_1 v_0 = u_{n-3} \ldots u_1 u_0$. Without loss of generality, we assume that $u$ is in $S^{0,1}$, and the degree of any node in the subgraph of $TQ_n$ induced by $S^{0,1}$ is $n - 2$. Let $\mathcal{N}(u) \cap S^{0,1} = \{w_3, w_4, \ldots, w_n\}$. Since $w_r \neq w_s$ for $3 \leq r \neq s \leq n$, we have $w_i^r \neq w_i^s$ for $(i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Case 1: $v_{n-1} v_{n-2} \neq u_{n-1} u_{n-2}$ and $P_{n-3}(u) = P_{n-3}(v) = 1$.

Subcase 1.1: $v_{n-1} v_{n-2} = u_{n-1} u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{0,0}$. Let $L_1$ be the path $u = u^{0,1}, v^{0,0} = v$, and $L_2$ be the path $u = u^{0,1}, u^{1,1}, u^{1,0}, v^{0,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Obviously, the length of $L_i$ is 3. If $P_{n-3}(w_i) = 0$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Obviously, the length of $L_i$ is 4. Since $P_{n-1}(u^{0,1}) \neq P_{n-1}(w_i^{0,0})$ or $P_{n-1}(w_i^{0,0}) \neq P_{n-1}(w_i^{0,0})$, $L_i$ is not a twisted path. Thus, we have $n$ disjoint paths joining $u$ to $v$ satisfying (1)-(3). See Fig. 2(a) for illustration.

Subcase 1.2: $v_{n-1} v_{n-2} = u_{n-1} u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,0}$. Let $L_1$ be the path $u = u^{0,1}, v^{0,0} = v$, and $L_2$ be the path $u = u^{0,1}, u^{1,1}, u^{1,0}, v^{0,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Thus, the length of $L_i$ is 3. If $P_{n-3}(w_i) = 0$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Thus, the length of $L_i$ is 4. Since $P_{n-1}(w_i^{1,0}) \neq P_{n-1}(w_i^{0,0})$ or $P_{n-1}(w_i^{0,0}) \neq P_{n-1}(w_i^{0,0})$, $L_i$ is not a twisted path. Thus, we have $n$ disjoint paths joining $u$ to $v$ satisfying (1)-(3). See Fig. 2(b) for illustration.

Subcase 1.3: $v_{n-1} v_{n-2} = u_{n-1} u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,1}$. Let $L_1$ be the path $u = u^{0,1}, v^{0,0} = v$, and $L_2$ be the path $u = u^{0,1}, u^{1,1}, u^{1,0}, v^{0,0} = v$. For each path $L_i$, with $3 \leq i \leq n$, set $L_i$ to be $u = u^{0,1}, w_i^{0,1}, w_i^{1,1}, v^{1,1} = v$. Thus, the length of $L_i$ is 3. It is observed that none of $L_i$ for $1 \leq i \leq n$ is a twisted path. We find $n$ disjoint paths joining $u$ to $v$ satisfying (1)-(3). See Fig. 2(c) for illustration.

Case 2: $v_{n-1} v_{n-2} \neq u_{n-1} u_{n-2}$ and $P_{n-3}(u) = P_{n-3}(v) = 0$.

Subcase 2.1: $v_{n-1} v_{n-2} = u_{n-1} u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{0,0}$. Set $L_1$ as $u = u^{0,1}, u^{1,0}, v^{0,0} = v$, and set $L_2$ as $u = u^{0,1}, u^{1,1}, u^{1,0}, v^{0,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Thus, the length of $L_i$ is 3. If $P_{n-3}(w_i) = 0$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{0,0}, v^{0,0} = v$. Thus, the length of $L_i$ is 4. We have constructed $n$ disjoint $(u, v)$-paths satisfying (1)-(3). See Fig. 3(a) for illustration.

Subcase 2.2: $v_{n-1} v_{n-2} = u_{n-1} u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,0}$. Set $L_1$ as $u = u^{0,1}, v^{1,0} = v$, and set $L_2$ as $u = u^{0,1}, u^{1,1}, u^{1,0}, v^{1,0} = v$. If $P_{n-3}(w_i) = 1$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{1,0}, w_i^{1,0}, v^{1,0} = v$. Thus, the length of $L_i$ is 4. If $P_{n-3}(w_i) = 0$ for $3 \leq i \leq n$, then set $L_i$ as $u = u^{0,1}, w_i^{0,1}, w_i^{1,0}, v^{1,0} = v$. Thus, the length of $L_i$ is 3. We have found $n$ disjoint $(u, v)$-paths satisfying (1)-(3). See Fig. 3(b) for illustration.
Fig. 2. Relative positions of the source node and the destination node in $R^2$, with $P_{i,j}(V) = 1$ for case 1.

(a) $u \in S_{0,1}^0$, $v \in S_{0,0}^0$

(b) $u \in S_{0,1}^1$, $v \in S_{0,1}^0$

(c) $u \in S_{0,1}^1$, $v \in S_{1,1}^1$
Fig. 3. Relative positions of the source node and the destination node in T(n) with $P_1(a) = P_2(b) = 0$ for case 2.
Subcase 2.3. \( v_{n-1}v_{n-2} = \bar{u}_{n-1}u_{n-2} \). In this case, \( u = u_{0,1} \) and \( v = v_{1,1} \). Let \( L_1 \) be the path \( u = u_{0,1}, v_{1,1} = v \), and \( L_2 \) be the path \( u = u_{0,1}, u_{0,0}, v_{1,1} = v \). For each \( L_i \) with \( 3 \leq i \leq n \), let \( L_i \) be the path \( u = u_{0,1}, w_{i,1}, v_{i,1}, v_{1,1} = v \). Thus, the length of \( L_i \) is 3. We have obtained \( n \) disjoint \((u,v)\)-paths satisfying (1)--(3). See Fig. 3(c) for illustration.

In cases 3--6, we consider \( u_{n-3}u_{n-4} \ldots u_0 \neq v_{n-3}v_{n-4} \ldots v_0 \). Since \( S_{0,1} \) induces a \( TQ_{n-2} \), by induction there are \( n-2 \) disjoint paths. Let \( L_{1,1}, L_{2,2}, \ldots, L_{n-2,1} \) be \( n-2 \) disjoint paths joining \( u_{0,1} \) to \( v_{0,1} \) such that (1) the length of each path is at most \( \lceil (n - 2)/2 \rceil + 2 \), (2) \( L_{0,1} \) is the shortest path joining \( u_{0,1} \) to \( v_{0,1} \) in \( S_{0,1} \), and (3) the length of \( L_{i,1} \) is at most \( \lceil (n - 2)/2 \rceil + 1 \) if \( L_{i,1} \) is a twisted path. Hence, the length of \( L_{i,1} \) is at least 2 if \( i > 1 \). Write \( L_{i,1} \) as \( u_{i,0,1}, u_{i,1}, \ldots, u_{i,k_i} = v_{0,1} \), where \( k_i \) is the length of \( L_{i,1} \). Let \( L_{i,j} \) be the corresponding path of \( L_{i,1} \) in \( S_{i,j} \) joining \( u_{i,j} \) to \( v_{i,j} \). Without loss of generality, we assume that \( u \) is in \( S_{0,1} \).

Case 3: \( u_{n-1}u_{n-2} = v_{n-1}v_{n-2} \). In this case, \( u = u_{0,1} \) and \( v = v_{0,1} \). We simply let \( L_i = L_{i,1} \) for \( 1 \leq i \leq n-2 \). We have obtained \( n-2 \) disjoint \((u,v)\)-paths satisfying (1)--(3). To construct the remaining two disjoint \((u,v)\)-paths \( L_{n-1} \) and \( L_n \), we consider the following three subcases.

Subcase 3.1. \( P_{n-3}(u) = P_{n-3}(v) = 1 \). Let \( L_{n-1} \) be the path \( u = u_{0,1}, u_{1,1} \ldots, u_{k,n-2} = v_{1,1} \), where \( k_i \) is the length of \( L_{i,1} \). Let \( L_{n} \) be the path \( u = u_{0,1} \). Since the length of \( L_{0,1} \) and \( L_{1,1} \) are at most \( \lceil (n - 2)/2 \rceil \), the length of \( L_{n-1} \) and \( L_n \) are at most \( \lceil (n - 2)/2 \rceil + 2 \). Note that \( \lceil (n - 2)/2 \rceil + 2 = \lceil n/2 \rceil + 1 \). See Fig. 4(a) for illustration.

Subcase 3.2. \( P_{n-3}(u) = P_{n-3}(v) = 0 \). Let \( L_{n-1} \) be the path \( u = u_{0,1}, u_{1,1} \ldots, u_{k,n-2} = v_{1,1} \), where \( 1 \leq i \leq n-2 \) with \( P_{n-3}(u) = 0 \). Let \( L_n \) be the path \( u = u_{0,1} \). Similarly, the length of \( L_{i,1} \) is at most \( \lceil (n - 2)/2 \rceil \). Therefore, the length of \( L_{n-1} \) and \( L_n \) are at most \( \lceil (n - 2)/2 \rceil + 2 \). See Fig. 4(b) for illustration.

Subcase 3.3. \( P_{n-3}(u) \neq P_{n-3}(v) \). Without loss of generality, we assume that \( P_{n-3}(u) = 0 \) and \( P_{n-3}(v) = 1 \). Let \( L_{n-1} \) be the path \( u = u_{0,1}, u_{1,1} \ldots, u_{k,n-2} = v_{1,1} \). Since \( L_{0,1} \) is not a twisted path and its length is at most \( \lceil n/2 \rceil + 2 \). Others paths \( L_{i,1} \) where \( 1 \leq i \leq n-2 \) with \( P_{n-3}(u) = 0 \) for all \( 1 \leq j < k_i \), set \( L_i \) as \( u = u_{i,0,1}, u_{i,1}, \ldots, u_{i,j} \ldots, u_{i,k_i} = v \). Since \( P_{n-1}(u_{i,j}) \neq P_{n-1}(u_{i,j}) \), \( L_i \) is not a twisted path and its length is at most \( \lceil n/2 \rceil + 2 \). We have constructed \( n-3 \) disjoint \((u,v)\)-paths satisfying (1)--(3). To construct the remaining three disjoint \((u,v)\)-paths \( L_1, L_{n-1} \) and \( L_n \), we consider the following three subcases.
Fig. 4. Relative positions of the source node and the destination node with $u, v$ in $S^v$ for case 3.
Subcase 4.1: $P_{n-3}(u) = P_{n-3}(v) = 1$. Let $L_1$ be $u = u^{L_1,1} \rightarrow v^{0,0}$, $v^{0,0} = v$, $L_{n-1}$ be $u = u^{0,1}, v^{0,0} = v$, and $L_n$ be $u = u^{0,1,1}, v^{0,0} = v$. See Fig. 5(a) for illustration.

Fig. 5. Relative positions of the source node and the destination node with $u$ in $S^{0,1}$ and $v$ in $S^{0,0}$ for case 4.
Fig. 5. (Continued)

(c) \( P_{n-3}(u) = P_{n-3}(v) = 0 \)

(d) \( P_{n-3}(u) \neq P_{n-3}(v) \)
Subcase 4.2: \( P_{n-3}(u) = P_{n-3}(v) = 0 \). If the length of \( L_{1}^{0,1} \) is 1, or \( P_{n-3}(u_{1,j}) = 0 \) for all \( 1 \leq j < k_{1} \), set \( L_{1} \) as \( u = u_{1,0}^{0,1} \rightarrow v_{1,1}^{0,1} \rightarrow v_{0,0}^{1,0} = v \), set \( L_{n-1} \) as \( u = u_{0,1}^{1,0} \rightarrow v_{1}^{0,0} = v \), and set \( L_{n} \) as \( u = u_{0,1}^{1,0} \rightarrow u_{1,1}^{0,0} \rightarrow v_{1,0}^{0,0} = v \). If the length of \( L_{1}^{0,1} \) is greater or equal to 2 with any node \( u_{1,j}^{0,1} \) satisfying \( P_{n-3}(u_{1,j}^{0,1}) = 1, 1 \leq j < k_{1} \), set \( L_{1} \) as \( u = u_{1,0}^{1,0}, u_{1,1}^{0,1}, \ldots, u_{1,j}^{0,1}, u_{i,j}^{1,0}, u_{1,1,0}^{0,1}, u_{1,1,0}^{0,1}, \ldots, v_{1,k_{1}}^{0,0} = v \), \( L_{n-1} \) as \( u = u_{0,1}^{1,0} \rightarrow u_{1,1}^{0,1} \rightarrow v_{1,0}^{0,0} = v \), and \( L_{n} \) as \( u = u_{0,1}^{1,0} \rightarrow u_{1,1}^{0,1} \rightarrow v_{1,0}^{0,0} = v \). See Fig. 5(b) and (c) for illustration.

Subcase 4.3: \( P_{n-3}(u) \neq P_{n-3}(v) \). Without loss of generality, we assume that \( P_{n-3}(u) = 0 \) and \( P_{n-3}(v) = 1 \). Let \( L_{1} \) be the path \( u = u_{0,1}^{0,1} \rightarrow v_{0,1}^{0,1} \rightarrow v_{0,0}^{1,0} = v \), \( L_{n-1} \) be the path \( u = u_{0,1}^{0,1} \rightarrow u_{1,0}^{1,0} \rightarrow v_{1}^{0,0} = v \), \( L_{n} \) be the path \( u = u_{0,1}^{0,1} \rightarrow u_{1,0}^{1,0} \rightarrow v_{1,0}^{0,0} = v \). See Fig. 5(d) for illustration. Hence we have constructed \( n \) disjoint \((u, v)\)-paths satisfying (1)-(3).

Case 5: \( v_{n-1}^{1,0} v_{n-2}^{1,0} = u_{n-1}^{1,0} u_{n-2}^{1,0} \). In this case, \( u = u_{0,1}^{0,1} \) and \( v = v_{1,0}^{1,0} \). For those paths \( L_{i}^{0,1} \) with any node \( u_{1,0}^{0,1} \) satisfying \( P_{n-3}(u_{1,0}^{0,1}) = 0 \), where \( 1 \leq i < n-2 \), \( 1 \leq j < k_{i} \), set \( L_{i} \) as \( u = u_{1,0}^{0,1} \rightarrow u_{1,1}^{0,1} \rightarrow u_{1,0}^{0,1} \rightarrow u_{1,1}^{0,1} \rightarrow \cdots \rightarrow u_{1,k_{i}}^{0,1} = v \). Obviously, the length of \( L_{i} \) is at most \( \lceil n/2 \rceil + 2 \). If \( L_{i} \) is a twisted path, \( L_{i}^{0,1} \) is a twisted path in \( S^{0,1} \). By induction, the length of \( L_{i}^{0,1} \) is at most \( \lceil (n-2)/2 \rceil + 1 \). This implies that the length of \( L_{i} \) is at most \( \lceil n/2 \rceil + 1 \) if \( L_{i} \) is a twisted path. Others paths \( L_{i}^{0,1} \) with \( P_{n-3}(u_{1,j}^{0,1}) = 1 \) for all \( 1 \leq i < k_{i} \), set \( L_{i} \) as \( u = u_{1,0}^{0,1}, u_{1,1}^{0,1}, u_{1,0}^{0,1}, u_{1,1}^{0,1}, \ldots, u_{1,k_{i}}^{0,1} = v \). It is easy to see that \( L_{i}^{0,1} \) is a twisted path in \( S^{0,1} \) with \( P_{n-3}(u_{1,0}^{0,1}) = 1 \) for all \( 1 \leq i < k_{i} \). Therefore, the length of \( L_{i} \) is at most \( \lceil n/2 \rceil + 2 \) and \( L_{i} \) is not a twisted path. We have found \( n-3 \) disjoint \((u, v)\)-paths satisfying (1)-(3). To construct the remaining three disjoint \((u, v)\)-paths \( L_{1}, L_{n-1}, \) and \( L_{n} \), we consider the following three subcases.

Subcase 5.1: \( P_{n-3}(u) = P_{n-3}(v) = 1 \). If the length of \( L_{1}^{0,1} \) is 1, or \( P_{n-3}(u_{1,j}) = 1 \) for all \( 1 \leq j < k_{1} \), set \( L_{1} \) as \( u = u_{0,1}^{0,1} \rightarrow v_{0,1}^{0,1} \rightarrow v_{0,0}^{1,0} = v \), \( L_{n-1} \) as \( u = u_{0,1}^{0,1} \rightarrow u_{1,0}^{1,0} \rightarrow v_{0,0}^{1,0} = v \), and \( L_{n} \) as \( u = u_{0,1}^{0,1} \rightarrow u_{1,1}^{0,1} \rightarrow u_{0,0}^{0,1} \rightarrow v_{0,0}^{1,0} = v \). If the length of \( L_{1}^{0,1} \) is greater or equal to 2 with any node \( u_{1,j}^{0,1} \) satisfying \( P_{n-3}(u_{1,j}^{0,1}) = 0, 1 \leq j < k_{1} \), let \( L_{1} \) be the path \( u = u_{0,1}^{0,1}, u_{1,0}^{0,1}, \ldots, u_{1,j}^{0,1}, u_{1,j}^{1,0}, u_{1,j}^{0,1}, \ldots, v_{1,k_{1}}^{0,1} = v \), \( L_{n-1} \) be the path \( u = u_{0,1}^{0,1}, u_{0,0}^{0,0} \rightarrow v_{0,0}^{1,0} \rightarrow v_{1,0}^{0,0} = v \), and \( L_{n} \) be the path \( u = u_{0,1}^{0,1}, u_{1,1}^{0,1} \rightarrow v_{1,0}^{0,0} = v \). See Fig. 6(a) and (b) for illustration.

Subcase 5.2: \( P_{n-3}(u) = P_{n-3}(v) = 0 \). Let \( L_{1} \) be the path \( u = u_{0,1}^{0,1} \rightarrow v_{0,0}^{1,0} \rightarrow v_{1,0}^{1,0} = v \), \( L_{n-1} \) be the path \( u = u_{0,1}^{0,1} \rightarrow u_{1,0}^{1,0} \rightarrow v_{1,0}^{1,0} = v \), and \( L_{n} \) be the path \( u = u_{0,1}^{0,1} \rightarrow u_{1,1}^{0,1} \rightarrow u_{0,0}^{0,0} \rightarrow v_{0,0}^{1,0} \rightarrow v_{1,0}^{1,0} = v \). If \( L_{n} \) is a twisted path, then \( L_{0,0}^{0,0} \) is a shortest path joining \( u_{0,0}^{0,0} \) to \( v_{0,0}^{1,0} \) satisfying \( P_{n-3}(z) = 0 \) for all nodes \( z \) in \( L_{0,0}^{0,0} \). It follows from Lemma 2 that the length of \( L_{0,0}^{0,0} \) is at most \( \lceil (n-2)/2 \rceil - 1 \). Hence, the
Fig. 6. Relative positions of the source node and the destination node with $u$ in $S^{0,1}$ and $v$ in $S^{1,0}$ for case 5.
(c) \( P_{n-3}(u) = P_{n-3}(v) = 0 \)

(d) \( P_{n-3}(u) \neq P_{n-3}(v) \)

Fig. 6. (Continued)
length of $L_n$ is at most $\left\lceil \frac{(n - 2)}{2} \right\rceil + 3 = \left\lceil \frac{n}{2} \right\rceil + 1$ if $L_n$ is a twisted path. See Fig. 6(c) for illustration.

**Subcase 5.3:** $P_{n-3}(u) \neq P_{n-3}(v)$. Without loss of generality, we assume that $P_{n-3}(u) = 0$ and $P_{n-3}(v) = 1$. Let $L_1$ be the path $u = u^{0,1}, u^{1,0}v^{0,1} = v$, $L_{n-1}$ be the path $u = u^{0,1}, u^{1,0}v^{1,0} = v$, and $L_n$ be the path $u = u^{0,1}, u^{1,1}v^{1,1}, v^{1,0} = v$. See Fig. 6(d) for illustration. Thus we have found $n$ disjoint $(u, v)$-paths satisfying (1)-(3).

**Case 6:** $v_{n-1}v_{n-2} = u_{n-1}u_{n-2}$. In this case, $u = u^{0,1}$ and $v = v^{1,1}$. Note that the length of $L_i^{0,1}$ is at least two for $1 < i < n - 2$. Set $L_i$ as $u = u^{0,1}, u^{i,1}, u^{i,1}, u^{i,2}, \ldots, u^{i,k} = v$ for $1 < i < n - 2$. Since $P_{n-1}(u^{0,1}) \neq P_{n-1}(u^{1,1})$, $L_i$ is not a twisted path and its length is at most $\left\lceil \frac{n}{2} \right\rceil + 2$. We have constructed $n - 3$ disjoint $(u, v)$-paths satisfying (1)-(3). To construct the remaining three disjoint $(u, v)$-paths $L_1, L_{n-1}$ and $L_n$, we consider the following three subcases.

**Subcase 6.1:** $P_{n-3}(u) = P_{n-3}(v) = 1$. Let $L_1$ be $u = u^{0,1}, v^{0,1} = v$, $L_{n-1}$ be $u = u^{0,1}, u^{1,1}v^{0,1} = v$, and $L_n$ be $u = u^{0,1}, u^{0,0}, u^{1,0}v^{1,0} = v$. Since $P_{n-1}(u^{0,1}) \neq P_{n-1}(u^{1,0})$, $L_n$ is not a twisted path and its length is at most $\left\lceil \frac{n}{2} \right\rceil + 2$. See Fig. 7(a) for illustration.

**Subcase 6.2:** $P_{n-3}(u) = P_{n-3}(v) = 0$. Let $L_1$ be $u = u^{0,1}, v^{0,1} = v$, $L_{n-1}$ be $u = u^{0,1}, u^{1,1}v^{1,1} = v$, and $L_n$ be $u = u^{0,1}, u^{0,0}, v^{0,0}v^{1,1} = v$. See Fig. 7(b) for illustration.

**Subcase 6.3:** $P_{n-3}(u) \neq P_{n-3}(v)$. Without loss of generality, we assume $P_{n-3}(u) = 0$ and $P_{n-3}(v) = 1$. Let $L_1$ be $u = u^{0,1}, v^{0,1}, v^{1,1} = v$, $L_{n-1}$ be $u = u^{0,1}, u^{1,1}v^{0,1}, v^{1,1} = v$, and $L_n$ be $u = u^{0,1}, u^{1,0}v^{1,0}, v^{1,1} = v$. See Fig. 7(c) for illustration. Hence we have constructed $n$ disjoint $(u, v)$-paths satisfying (1)-(3). □

The following corollary follows from Lemma 4 and that the degree of each vertex in $TQ_n$ is $n$.

**Corollary 1.** Assume $n$ is an odd integer. The connectivity of $TQ_n$, $\kappa(TQ_n)$, is $n$, and $D_\kappa(TQ_n) \leq \left\lceil \frac{n}{2} \right\rceil + 2$. Hence, $TQ_n$ is maximal connection.

The following theorem follows from Lemmas 3 and 4.

**Theorem 3.** $D_{n-1}(TQ_n) = D_n(TQ_n) = \left\lceil \frac{n}{2} \right\rceil + 2$ if $n$ is odd.

4. Embedding of cycles

A cycle structure is often used as a connection structure for local area network, for example Token Rings, and can also be used as a control/data flow
Fig. 7. Relative positions of the source node and the destination node with $u$ in $S^{0,1}$, $v$ in $S^{1,1}$ for case 6.
structure for distributed computations in arbitrary networks. In this section, we will show that $TQ_n$ contains a cycle $C_i$ of length $i$ for all $4 \leq i \leq 2^n$.

**Theorem 4.** Let $n$ be an odd integer and $n \geq 3$. For all $i$ with $4 \leq i \leq 2^n$, there exists a cycle $C_i = (u^0, u^1, \ldots, u^{i-1}, u^0)$ of length $i$, where $u^0 = 0^n$, $u^{i-1} = 0^{n-1}120^{i-2}$ and $t$ is an odd integer with $3 \leq t \leq n$ such that $2t^2 < i \leq 2t'$.

**Proof.** We prove this lemma by induction. In $TQ_3$, we have the following cycles:

$$
C_4 = (000, 100, 010, 110, 000), \\
C_5 = (000, 001, 011, 010, 110, 000), \\
C_6 = (000, 100, 010, 111, 111, 110, 000), \\
C_7 = (000, 100, 101, 001, 011, 111, 110, 000), \\
C_8 = (000, 001, 011, 010, 100, 101, 111, 110, 000).
$$

With these five cycles, it is easy to see that the lemma is true for $n = 3$. Assume that the Lemma is true for any odd $k$ with $3 \leq k < n$.

For $4 \leq i \leq 2^{n-2}$, by induction there exists a cycle $C_i = (u^0, u^1, \ldots, u^{i-1}, u^0)$ of length $i$ in $TQ_{n-2}$ where $u^0 = 0^{n-2}, u^{i-1} = 0^{n-1}120^{i-2}$ and $t$ is an odd integer with $3 \leq t \leq n-2$ such that $2t^2 < i \leq 2t$. Since $S^{0,0}$ induces $TQ_{n-2}$, $TQ_n$ contains a cycle $C_i$ of length $i$ for all $4 \leq i \leq 2^n-2$ in $S^{0,0}$ where $u^0 = 0^n, u^{i-1} = 0^{n-1}120^{i-2}$ and $t$ is an odd integer with $3 \leq t \leq n$ such that $2t^2 < i \leq 2t$. We first consider $2^n-2 < i \leq 2^{n-1}$. Then there exist two integers $a, b$ such that $a + b = i$ and $2^{n-3} < a, b < 2^{n-2}$. By induction, in $TQ_{n-2}$ there exist two cycles $C_a = (u_0^a, \ldots, u_{a-1}^a, u_0^a)$ and $C_b = (v_0^b, \ldots, v_{b-1}^b, v_0^b)$ with $u^0 = v^0 = 0^{n-2}$ and $u^{a-1} = v^{b-1} = 10^{n-4}$. Let $C_0^0 = (x_0, x_1, \ldots, x_{a-1}, x_0)$ denote the corresponding cycle of $C_a$ in $S^{0,0}$ and $C_0^1 = (y_0, y_1, \ldots, y_{a-1}, y_0)$ denote the corresponding cycle of $C_b$ in $S^{1,1}$. Obviously, $x^0 = 0^n, x^{a-1} = 0^2120^{n-4}, y^0 = 120^{n-2}$ and $y^{b-1} = 140^{n-4}$. We define $z^j = x^j$ if $0 \leq j \leq a-1$ and $z^j = y^{(i-j-1)}$ if $a \leq j \leq i-1$. It is easy to see that $(z^0, z^1, \ldots, z^{i-1}, z^0)$ forms a cycle of length $i$ such that $z^0 = 0^n$ and $z^{i-1} = 120^{n-2}$.

Now we consider $2^{n-1} < i \leq 2^n$. Then there exist four integers $a, b, c, d$ such that $a + b + c + d = i$ and $2^{n-3} < a, b, c, d < 2^{n-2}$. By induction, in $TQ_{n-2}$ there exist four cycles $C_a = (p_0^a, \ldots, p_{a-1}^a, p_0^a), C_b = (q_0^b, \ldots, q_{b-1}^b, q_0^b), C_c = (r_0^c, \ldots, r_{c-1}^c, r_0^c)$, and $C_d = (s_0^d, \ldots, s_{d-1}^d, s_0^d)$ with $p^0 = q^0 = r^0 = s^0 = 0^{n-2}$ and $p^{a-1} = q^{b-1} = r^{c-1} = s^{d-1} = 120^{n-4}$. Let $C_0^0 = (u_0^0, \ldots, u_{a-1}^0, u_0^0), C_0^1 = (v_0^1, \ldots, v_{b-1}^1, v_0^1), C_0^2 = (x_0^2, \ldots, x_{c-1}^2, x_0^2)$ and $C_0^3 = (y_0^3, \ldots, y_{d-1}^3, y_0^3)$ denote the corresponding cycles of $C_a, C_b, C_c, C_d$ in $S^{0,0}, S^{1,0}, S^{0,1}$ and $S^{1,1}$, respectively. Obviously, $u^0 = 0^n, u^{a-1} = 0^2120^{n-4}, v^0 = 10^{n-1}, v^{b-1} = 10120^{n-4}, x^0 = 010^{n-2}, x^{c-1} = 01120^{n-4}$, $y^0 = 120^{n-2}$ and $y^{d-1} = 140^{n-4}$. We define $z^j = u^j$ if $0 \leq j \leq a-1$, $z^j = v^{(a+b-j-1)}$ if $a \leq j \leq a + b - 1$, $z^j = x^{(j-a-b)}$ if $a + b \leq j \leq c-1$, $z^j = y^{(i-j-1)}$ if $c \leq j \leq c + d - 1$, and $z^j = z^{(i-j-1)}$ if $c + d \leq j \leq i-1$. It is easy to see that $(z^0, z^1, \ldots, z^{i-1}, z^0)$ forms a cycle of length $i$ such that $z^0 = 0^n$ and $z^{i-1} = 120^{n-2}$.
a + b + c - 1 and \( z^j = y^{(i-j-1)} \) if \( a + b + c \leq j \leq i - 1 \). It is easy to see that 
\( \langle z^0, z^1, \ldots, z^{i-1}, z^0 \rangle \) forms a cycle of length \( i \) such that \( z^0 = 0^n \) and \( z^{i-1} = 1^20^{n-2} \).

Based on the above proof idea, we can easily construct a cycle of arbitrary length. We here illustrate two examples of constructing \( C_{11} \) and \( C_{21} \) in \( TQ_5 \). To construct \( C_{11} \) in \( TQ_5 \), we find two cycles of length 6 and 5 in \( TQ_3 \), where \( C_6 \) and \( C_5 \) are given by

\[
C_6 = \langle 000, 100, 010, 011, 111, 110, 000 \rangle,
\]

\[
C_5 = \langle 000, 001, 011, 010, 110, 000 \rangle,
\]

then \( C_6^{0,0} \) and \( C_5^{1,1} \) are

\[
C_6^{0,0} = \langle 00000, 00100, 00010, 00011, 00111, 00110, 00000 \rangle,
\]

\[
C_5^{1,1} = \langle 11000, 11001, 11011, 11010, 11110, 11000 \rangle,
\]

respectively. We can use \( C_6^{0,0} \) and \( C_5^{1,1} \) to construct a \( C_{11} \) as follows:

\[
C_{11} = \langle 00000, 00100, 00010, 00011, 00111, 00110, 01110, 11010, 11011, 11001, 11000, 00000 \rangle.
\]

Similarly, we can construct a \( C_{21} \) as follows:

\[
C_{21} = \langle 00000, 00100, 00010, 00011, 00111, 00110, 01110, 10110, 10010, 10011, 10001, 10000, 01000, 01001, 01011, 01010, 01110, 11110, 11010, 11011, 11001, 11000, 00000 \rangle.
\]

5. Concluding remarks

This paper studies wide diameter, fault diameter and embedding cycles problems in twisted cubes. We have shown that the twisted cube is to improve the performance of the hypercube. It is known that \( D_{n-1}(Q_n) = D_n(Q_n) = n + 1 \). In this paper, we have proven that wide diameter and fault diameter of the twisted cube are about the half of the corresponding parameters of the hypercube. Furthermore, we also proved that the twisted cube is a pancyclic network. Hence, the twisted cube is an attractive topology for interconnection networks.
References