The Consecutive-4 Digraphs are Hamiltonian

Gerard J. Chang, Frank K. Hwang, and Li Da Tong

DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL CHIAO TUNG UNIVERSITY
HSINCHU 30050, TAIWAN

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Abstract: Du, Hsu, and Hwang conjectured that consecutive-d digraphs are Hamiltonian for \( d = 3, 4 \). Recently, we gave an infinite class of consecutive-3 digraphs, which are not Hamiltonian. In this article we prove the conjecture for \( d = 4 \).

Keywords: Hamiltonian circuit; consecutive-d digraph; network; loop

1. INTRODUCTION

Define \( G(d, n, q, r) \), also known as a consecutive-d digraph, to be a digraph whose \( n \) nodes are labeled by the residues modulo \( n \), and a link \( i \rightarrow j \) from node \( i \) to node \( j \) exists if and only if \( j \in \{ qi + k \pmod{n} : r \leq k \leq r + d - 1 \} \), where \( 1 \leq q \leq n - 1, 1 \leq d \leq n - 1 \) and \( 0 \leq r \leq n - 1 \) are given. Many computer networks and multiprocessor systems use consecutive-d digraphs for the topology of their interconnection networks. For example, \( q = 1 \) yields the multiloop networks [13], also known as circulant digraphs [14], with the skip set \( \{ r, r + 1, \ldots, r + d - 1 \} \). \( q = d \) and \( r = 0 \) yields the generalized de Bruijn digraphs [8, 12], and \( q = r = n - d \) yields the Imase–Itoh digraphs [9].

In some applications, it is important to know whether a consecutive-d digraph embeds a Hamiltonian circuit. This issue was first raised by Pradhan [11]. Necessary and sufficient conditions for generalized de Bruijn digraphs and the Imase–Itoh digraphs to be Hamiltonian were given by Du, Hsu, Hwang, and Zhang [5]. For
the case of \( \gcd(n, q) \geq 2 \), Du, Hsu, and Hwang [4] showed that \( G(d, n, q, r) \) is Hamiltonian if and only if \( d \geq \gcd(n, q) \). So, we may only consider the case when \( \gcd(n, q) = 1 \). Necessary and sufficient conditions for consecutive-\( d \) digraphs to be Hamiltonian were given by Hwang [7] for \( d = 1 \) and by Du and Hsu [3] (also see [2]) for \( d = 2 \). Furthermore, Du, Hsu, and Hwang [4] proved that consecutive-\( d \) digraphs are Hamiltonian for \( d \geq 5 \), and conjectured they are also for \( d = 3, 4 \). Du and Hsu [3] gave partial support to this conjecture by proving its validity under the condition \( q \leq d \). Recently, we [1] gave an infinite class of examples that consecutive-3 digraphs are not necessarily Hamiltonian. In this article, we prove that consecutive-4 digraphs are Hamiltonian, and thus, completely settle the conjecture.

2. SOME GENERAL REMARKS

Throughout this article, we assume that \( \gcd(n, q) = 1 \). In this case, \( G(d, n, q, r) \) is a regular digraph of indegree and outdegree both \( d \). In particular, \( G(1, n, q, r') \) is the disjoint union of cycles.

Let \( G(4, n, q, r) \) denote the underlying consecutive-4 digraph. Consider the digraph \( G(1, n, q, r + 1) \). Suppose that \( G(1, n, q, r + 1) \) consists of \( c \) disjoint cycles \( C_1, C_2, \ldots, C_c \). If \( c = 1 \), then \( G(4, n, q, r) \) is Hamiltonian and we are done. Suppose that \( c > 1 \). A link-interchange method was introduced in [4] to merge two cycles. Since \( 0 < q < n \), there exists a cycle with more than one node. Furthermore, this cycle remains to contain more than one node throughout merges. Let \( i \) be a node on this cycle such that \( i + 1 \) is not. Such an \( i \) always exists unless the cycle is Hamiltonian. Suppose that \( i' \rightarrow i \) and \( (i + 1)' \rightarrow i + 1 \) are in \( G(1, n, q, r + 1) \), where \( i' \neq i \) but \( (i + 1)' \) could be \( i + 1 \). We replace these two links by the two links \( i' \rightarrow i + 1 \) and \( (i + 1)' \rightarrow i \) and call this an \( \{i, i + 1\} \) interchange, which merges the two cycles \( i \) and \( i + 1 \) are on into one. Note that the link \( i' \rightarrow i + 1 \) is in \( G(1, n, q, r + 2) \) and the link \( (i + 1)' \rightarrow i \) is in \( G(1, n, q, r) \).

Two interchanges \( \{i, i + 1\} \) and \( \{j, j + 1\} \) do not interfere with each other, if \( \{i, i + 1\} \cap \{j, j + 1\} = \emptyset \). But if the intersection is not empty, say, \( j + 1 = i \), then doing the interchange \( \{i - 1, i\} \) after \( \{i, i + 1\} \) means replacing \( (i + 1)' \rightarrow i \) by \( (i + 1)' \rightarrow i - 1 \), which is in \( G(1, n, q, r - 1) \), but not in \( G(4, n, q, r) \). However, we can do \( \{i, i + 1\} \) after \( \{i - 1, i\} \). This is because we are replacing \( (i - 1)' \rightarrow i \) and \( (i + 1)' \rightarrow i + 1 \) by \( (i - 1)' \rightarrow i + 1 \) and \( (i + 1)' \rightarrow i \), where \( (i - 1)' \rightarrow i + 1 \) is in \( G(1, n, q, r + 3) \) and \( (i + 1)' \rightarrow i \) is in \( G(1, n, q, r) \). Therefore, we can do two consecutive interchanges, if we do it in the right order, namely, do the smaller pair first. Similarly, if we start with decomposing \( G(1, n, q, r + 2) \) into cycles, then we can do two consecutive interchanges, if we do the larger pair first.

We will now represent two consecutive interchanges \( \{i - 1, i\} \) and \( \{i, i + 1\} \) by the set \( \{i - 1, i, i + 1\} \). In defining an interchange, the two nodes involved are assumed to be on different cycles, and these are cycles updated to previous merges. For example, when the interchange \( \{i, i + 1\} \) is performed after the interchange
\{i - 1, i\}, then the three nodes \(i - 1, i, i + 1\) are on different cycles originally (if \(i - 1\) and \(i + 1\) are on the same cycle, we have no reason to perform the second interchange). A legitimate interchange set (without three consecutive interchanges) can be represented by a set \(S = \{S_1, S_2, \ldots, S_b\}\), where the \(S_i\)'s are disjoint and each \(S_i\) is a subset of two or three consecutive nodes. Note that after one or more interchanges in \(S_i\) are performed, then all cycles intersecting \(S_i\) are connected.

Let \(X\) and \(Y\) be two sets of subsets of \(\{1, 2, \ldots, m\}\). Define \(B_m(X, Y)\) to be the bipartite graph with vertex set \(X \cup Y\), and there exists an edge between \(X_i \in X\) and \(Y_j \in Y\) if and only if \(X_i \cap Y_j \neq \emptyset\). Let \(C^r+1\) (respectively, \(C^r+2\)) denote the set of all disjoint cycles in \(G(1, n, q, r + 1)\) (respectively, \(G(1, n, q, r + 2)\)). Then we have the following.

**Lemma 1.** \(G(4, n, q, r)\) is Hamiltonian if \(gcd(n, q) = 1\) and there exists a legitimate interchange set \(S\) such that either \(B_n(S, C^r+1)\) or \(B_n(S, C^r+2)\) is connected.

**Proof.** Since \(gcd(n, q) = 1\), both \(G(1, n, q, r + 1)\) and \(G(1, n, q, r + 2)\) are disjoint unions of cycles. Applying the link-interchange method by using the legitimate interchange set \(S\), we can merge the cycles into a Hamiltonian cycle of \(G(4, n, q, r)\). Q.E.D.

### 3. Algorithm for Constructing \(S\)

We are unable to find an explicit legitimate interchange set \(S\) such that \(B_n(S, C^r+1)\) or \(B_n(S, C^r+2)\) is connected for all \(n\) and \(q\). However, for each given set \((n, q, r)\), we give an algorithm to construct such \(S\). In fact, our construction applies to a more general setting where \(C_1, C_2, \ldots, C_r\) do not have to come from \(G(1, n, q, r + 1)\) or \(G(1, n, q, r + 2)\), but merely a disjoint partition of \(\{1, 2, \ldots, n\}\).

**Lemma 2.** Let \(P = \{P_1, P_2, \ldots, P_p\}\) be a partition of \(\{1, 2, \ldots, m\}\) such that all \(|P_j| \geq 2\) except one part can be a singleton. Then there exists \(S = \{S_1, S_2, \ldots, S_b\}\), where \(S_i\)'s are disjoint consecutive subsets of \(\{1, 2, \ldots, m\}\) with all \(|S_i| = 2\) or \(3\) such that \(B_m(S, P)\) is connected and the \(S_i\) containing \(m\) (if any) has \(|S_i| = 2\).

**Proof.** We shall prove the lemma by induction on \(m\). It is trivially true for \(m \leq 4\). Assume \(m \in P_i\) and \(m - 1 \in P_j\).

If \(|P_i| \geq 3\), then \(|P_i - \{m\}| \geq 2\). Let \(P'\) be obtained from \(P\) by deleting \(m\) from \(P_i\). By the induction hypothesis, there exists \(S\) such that \(B_{m-1}(S, P')\) is connected. Clearly, \(B_m(S, P)\) is also connected.

Now, suppose that \(|P_i| \leq 2\). If \(i \neq j\), let \(P'\) be obtained from \(P\) by replacing \(P_i\) and \(P_j\) by \(P' = P_i \cup P_j - \{m - 1, m\}\). Note that \(P'\) is nonempty, since \(P_i\) or \(P_j\) is not a singleton. Also, \(P'\) is a singleton only when \(P_i\) or \(P_j\) is. Thus, \(P'\) has at most one singleton. By the induction hypothesis, there exists \(S'\) such that \(B_{m-2}(S', P')\) is connected. Then \(B_m(S' \cup \{\{m - 1, m\}\}, P)\) is connected.

If \(i = j\), i.e., \(P_i = \{m - 1, m\}\), let \(P' = P - \{P_i\}\). By the induction hypothesis, \(B_{m-2}(S', P')\) is connected for some \(S'\). Set \(S = S' \cup \{\{m - 2, m - 1\}\}\), if \(m - 2\)
is not in any $S_k$. Otherwise, assume $m - 2 \in S_k$ (then $|S_k| = 2$). Let $S$ be obtained from $S'$ by adding $m - 1$ to $S_k$. Then $B_{m}(S, P)$ is connected. Q.E.D.

**Theorem 1.** Suppose that $\gcd(n, q) = 1$. Then $G(4, n, q, r)$ is Hamiltonian.

**Proof.** We first note that a consecutive-1 digraph $G(1, n, q, r')$ has a loop $i \rightarrow i$ (i.e., $i \equiv qi + r' (\mod n)$) if and only if $\gcd(n, q - 1)$ divides $r'$. In the affirmative case, the number of loops is $\gcd(n, q - 1)$, see [7].

If $\gcd(n, q - 1) > 1$, then either $G(1, n, q, r + 1)$ or $G(1, n, q, r + 2)$ has no loop, as $\gcd(n, q - 1)$ cannot divide both $r + 1$ and $r + 2$. If $\gcd(n, q - 1) = 1$, then both $G(1, n, q, r + 1)$ and $G(1, n, q, r + 2)$ have exactly one loop. In either case, since $\gcd(n, q) = 1$, either $G(1, n, q, r + 1)$ or $G(1, n, q, r + 2)$ partitions the node-set into a set $C$ of disjoint cycles with at most one singleton-cycle. By Lemma 2, there exists a legitimate interchange set $S$ such that $B_{n}(S, C)$ is connected. The theorem then follows from Lemma 1.

Q.E.D.

Note that the inductive proof of Lemma 2 implies a linear-time algorithm to construct $S$.

**4. EXPLICIT CONSTRUCTION OF $S$**

When $\gcd(n, q) = 1$ and 3 divides $n$, we can give an explicit construction of $S$ that works for all $n$ and $q$. Throughout this section, $S = \{3i - 2, 3i - 1, 3i : i = 1, 2, \ldots, n/3\}$.

**Theorem 2.** If $\gcd(n, q) = 1$ and 3 divides $n$, then either $B_{n}(S, C^{r + 1})$ or $B_{n}(S, C^{r + 2})$ is connected.

**Proof.** It is now easier to consider $S$ as a set $E$ of links (a subset of size 3 corresponds to two consecutive links). To show $B_{n}(S, C^{r + 1})$ or $B_{n}(S, C^{r + 2})$ is connected, it suffices to show that $E \cup G(1, n, q, r + 1)$ or $E \cup G(1, n, q, r + 2)$ is connected. We first consider $E \cup G(1, n, q, r + 1)$. Note that for $i \rightarrow i + 1$ in $E$, both $i \rightarrow qi + r + 1$ and $i + 1 \rightarrow q(i + 1) + r + 1$ are in $G(1, n, q, r + 1)$. Hence, $qi + r + 1$ and $qi + r + 1 + q$ are connected in $E \cup G(1, n, q, r + 1)$. Let $E \cup Q$ be obtained from $E \cup G(1, n, q, r + 1)$ by replacing the two links $i \rightarrow qi + r + 1$ and $i + 1 \rightarrow q(i + 1) + r + 1$ with the $q$-link $qi + r + 1 \rightarrow qi + r + 1 + q$ for every $i$ such that $i \rightarrow i + 1$ is in $E$. Then $E \cup G(1, n, q, r + 1)$ is connected if $E \cup Q$ is. We now explore the connectivity of $E \cup Q$.

Partition the nodes into $n/3$ groups, where group $i$ consists of nodes $3i - 2, 3i - 1, 3i$. We will refer to them as the first, second, and third node of the group. We show that the groups are interconnected through the $q$-links. A $q$-link $(i, j)$ will be called an $(x, y)$ $q$-link if $i$ is the $x^{th}$ node of a group and $j$ the $y^{th}$ node of a group. Since $\gcd(n, q) = 1$ and 3 divides $n$, we have that 3 does not divide $q$. Therefore, each group has two $q$-links going out and two $q$-links going in. The $2n/3$ $q$-links contain two patterns of size $n/3$ each: one pattern corresponds to the $(x, y)$ pattern of the $q$-link generated by the link $(1, 2)$, the other by the link $(2, 3)$. As 3 does not divide $q$, we have $x \neq y (\mod 3)$. So there are six permissible combinations for
these two patterns: (i) (1, 2), (2, 3); (ii) (1, 3), (3, 2); (iii) (2, 3), (3, 1); (iv) (2, 1), (1, 3); (v) (3, 1), (1, 2); (vi) (3, 2), (2, 1). The two q-links \((i, j)\) and \((i', j')\) going out from a group have different patterns \((x, y)\) and \((x', y')\). Note that \(i - j = i' - j'\).

Since \(i\) and \(i_0\) are in the same group, \(j\) and \(j_0\) are either in the same group or in consecutive groups. Furthermore, it is easily seen that \(j\) and \(j_0\) are in the same group if and only if \((x - y)(x' - y') > 0\). Thus, for the middle four combinations, the two q-links from a group go to two consecutive groups. This implies that every pair of consecutive groups is connected; hence \(E \cup Q\) is.

For the first and last pattern, the two q-links from a group go to the same group. So \(E \cup Q\) is not connected. However, let \(E \cup Q'\) be obtained from \(E \cup G(1, n, q, r + 2)\) by replacing the two links \(i \rightarrow qi + r + 2\) and \(i + 1 \rightarrow q(i + 1) + r + 2\) with the q-link \(qi + r + 2 \rightarrow qi + r + 2 + q\) for every \(i\) such that \(i \rightarrow i + 1\) is in \(E\). Then the combination of the two patterns of q-links is (2, 3) for case (i), and (1, 3), (3, 2) for case (vi). In either case, the two q-links from a group go to two different groups. So, \(E \cup Q'\), consequently, \(E \cup G(1, n, q, r + 2)\) is connected. Q.E.D.

Unfortunately, \(E = \{(3i - 2 \rightarrow 3i - 1) \cup (3i - 1 \rightarrow 3i) : i = 1, 2, \ldots, \lfloor n/3 \rfloor\}\) does not work when 3 does not divide \(n\). A counterexample \(G(4, 25, 13, 7)\) was given by Xuding Zhu (group 1 and group 5 are not connected in \(G(1, 25, 13, 8)\) and group 4 and group 8 not connected in \(G(1, 25, 13, 9)\)).

5. CONCLUSIONS

It is known that consecutive-\(d\) digraph is Hamiltonian for \(d \geq 5\), but not necessarily so for \(d \leq 3\). In this article, we prove the conjecture that consecutive-4 digraphs are Hamiltonian, and thus completely settle the issue. Of course, our result for \(d = 4\) implies that for \(d \geq 5\). Our result also implies that there exist at least \(\lfloor d/4 \rfloor\) disjoint Hamiltonian circuits for a consecutive-\(d\) digraph.

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References

