

controllers can be obtained to attenuate external perturbations which are bounded to a ball and have a uniform distribution. Similar results are expected in robust control, maximum likelihood parameter estimation, etc.

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Stability Analysis and Bang–Bang Sliding Control of a Class of Single-Input Bilinear Systems

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Abstract—This paper introduces a novel bang–bang sliding control of a class of single-input bilinear systems. The sliding function is chosen via the well-known pole-assignment method for linear time-invariant systems. Importantly, the bang–bang sliding control generates a reaching-and-sliding region and a stable-sliding region, each expressed by a set of linear inequalities. Both regions comprise the equilibrium point, shown to be asymptotically stable. However, the stability analysis is processed under the limitation that the system state should be initially located in the reaching-and-sliding region. Two numeric examples are used for demonstration.

Index Terms—Bang–bang sliding control, bilinear systems, stability analysis.

I. INTRODUCTION

In general, bilinear systems are expressed by a state differential equation, which is linear in control and linear in state but not jointly linear in state and control. Bilinear systems have been found in diverse processes and fields (see [1] and [2] for an excellent introduction) and many control strategies have developed, such as the quadratic feedback control [3], [4] and the sliding-mode control [5], [6]. Here, we will focus on a class of "bang–bang" single-input bilinear systems with controller designed by the sliding-mode theory [7], [8]. Note that the term "bang–bang" means the single input only switches between two fixed values.

Further assume the bilinear system is time invariant and controllable while the multiplicative terms of the single input and state variables are omitted, i.e., the bilinear system can be treated as a linear time-invariant (LTI) system by neglecting all these multiplicative terms. Based on this LTI system, a novel design technique of sliding function, called the pole-assignment-based method, is presented by directly using the prevailing pole-assignment method [9]. Actually, there are many other techniques for the sliding function design, such as the eigenstructure-assignment method [10] and the Lyapunov-based method [11]. For an LTI system, the Lyapunov-based method is simple and quite straightforward to derive a sliding function, but it is not suitable for a bilinear system due to the fact that the existing multiplicative terms still disturb the system behavior and thus, complicate the process of stability analysis. As for the eigenstructure-assignment method, its main idea is to generate a desired eigenstructure of the sliding mode, just like the pole-assignment-based method introduced in this paper. Therefore, the eigenstructure-assignment method can be also found useful for a bilinear system; however, it is often difficult to achieve an appropriate eigenstructure via the eigenstructure-assignment method. The pole-assignment-based method is then proposed to make it easier to determine the eigenstructure of the sliding mode. In addition, with the help of well-developed tools such as The MATLAB software, the process of the sliding function design is highly simplified in this method.

Since the bang–bang control only can switch between two finite values, the system should be finally restricted to a bounded area. In fact, this area relates to two important regions, called the reaching-and-

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sliding (RAS) region and the stable-sliding (SS) region. Each region is formulated by a set of linear inequalities. When a system is initially located in the RAS-region, its trajectory will reach the SS-region and then move toward the origin.

In Section II, the bang–bang sliding control is introduced. Also discussed is the stabilizability to the origin by means of reachability conditions to sliding mode and stability on the designed manifold. Section III uses two numeric examples of second-order bilinear systems to demonstrate the bang–bang sliding control in simulations [12]. Finally, concluding remarks are given in Section IV.

II. DESIGN OF BANG–BANG SLIDING CONTROL AND STABILITY ANALYSIS

General bilinear systems are mathematically expressed by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \sum_{k=1}^m \mathbf{N}_k \mathbf{x} u_k \quad (1)$$

where $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^T$ is the state, $\mathbf{u} = [u_1 \ u_2 \ \cdots \ u_m]^T$ is the control, and $\mathbf{A} (\in R^{n \times n})$, $\mathbf{B} (\in R^{n \times m})$, $\mathbf{N}_k (\in R^{n \times n})$ are the system matrices. This paper will focus on the case of single input, described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u + \mathbf{N}\mathbf{x}u \quad (2)$$

where $\mathbf{b} \in R^n$ and $u \in \{1, -1\}$. Note that u switches between 1 and -1 depending on a switching condition concerning the system state \mathbf{x} . Besides, the system is assumed controllable when $\mathbf{N}\mathbf{x}u$ is neglected. Hence, a gain vector $\mathbf{k} (\in R^n)$ can be uniquely obtained from the pole-assignment method by assigning n eigenvalues to $\mathbf{A} - \mathbf{b}\mathbf{k}^T$. The objective is to design a sliding controller for the single-input bilinear system (2).

Now, let us introduce the pole-assignment-based method to design the sliding function. First, the n eigenvalues λ_i , $i = 1, 2, \dots, n$, of $\mathbf{A} - \mathbf{b}\mathbf{k}^T$ are selected to satisfy:

- C1) All the eigenvalues are real and negative, i.e., $\lambda_i < 0$, $i = 1, 2, \dots, n$;
- C2) All the eigenvalues are distinct, i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$;
- C3) The n th eigenvalue λ_n is not in the spectrum of \mathbf{A} .

Based on these conditions, we have

$$(\mathbf{A} - \mathbf{b}\mathbf{k}^T) [\mathbf{W}_{n-1} \ \mathbf{w}_n] = [\mathbf{W}_{n-1} \ \mathbf{w}_n] \begin{bmatrix} \mathbf{\Lambda}_{n-1} & \mathbf{0} \\ \mathbf{0}^T & \lambda_n \end{bmatrix} \quad (3)$$

where

$$\mathbf{\Lambda}_{n-1} = \text{diag}[\lambda_1 \ \cdots \ \lambda_{n-1}], \quad \mathbf{W}_{n-1} = [\mathbf{w}_1 \ \cdots \ \mathbf{w}_{n-1}]$$

and \mathbf{w}_i is the i th right-eigenvector corresponding to λ_i , for $i = 1, 2, \dots, n$. Let

$$\begin{bmatrix} \mathbf{V}_{n-1}^T \\ \mathbf{v}_n^T \end{bmatrix} = [\mathbf{W}_{n-1} \ \mathbf{w}_n]^{-1} \quad \text{and} \quad \mathbf{V}_{n-1} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_{n-1}]$$

then (3) becomes

$$\begin{bmatrix} \mathbf{V}_{n-1}^T \\ \mathbf{v}_n^T \end{bmatrix} (\mathbf{A} - \mathbf{b}\mathbf{k}^T) = \begin{bmatrix} \mathbf{\Lambda}_{n-1} & \mathbf{0} \\ \mathbf{0}^T & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{V}_{n-1}^T \\ \mathbf{v}_n^T \end{bmatrix} \quad (4)$$

where \mathbf{v}_i^T is the i th left-eigenvector. Rearranging the second row of (4) leads to

$$\mathbf{v}_n^T (\mathbf{A} - \lambda_n \mathbf{I}_n) = (\mathbf{v}_n^T \mathbf{b}) \mathbf{k}^T \quad (5)$$

where \mathbf{I}_n is the $n \times n$ identity matrix. From C3), $\mathbf{A} - \lambda_n \mathbf{I}_n$ is nonsingular and thus, $\mathbf{v}_n^T \mathbf{b} \neq 0$. It results in

$$(\mathbf{v}_n^T \mathbf{b})^{-1} \mathbf{v}_n^T = \mathbf{k}^T (\mathbf{A} - \lambda_n \mathbf{I}_n)^{-1}. \quad (6)$$

Let the sliding function be chosen as

$$s = \mathbf{c}^T \mathbf{x} = \mathbf{k}^T (\mathbf{A} - \lambda_n \mathbf{I}_n)^{-1} \mathbf{x} \quad (7)$$

i.e., $\mathbf{c}^T = \mathbf{k}^T (\mathbf{A} - \lambda_n \mathbf{I}_n)^{-1}$, which also results in

$$\mathbf{c}^T \mathbf{A} = \lambda_n \mathbf{c}^T + \mathbf{k}^T. \quad (8)$$

From (6) we have $\mathbf{c}^T \mathbf{b} = 1$. Then the derivative of s with respect to time becomes

$$\dot{s} = \lambda_n s + \mathbf{k}^T \dot{\mathbf{x}} + (1 + \mathbf{c}^T \mathbf{N}\mathbf{x}) u. \quad (9)$$

It is reasonable to assume $1 + \mathbf{c}^T \mathbf{N}\mathbf{x} > 0$ for the system state \mathbf{x} near the origin. Let the control bang–bang switch between 1 and -1 as

$$u = -\text{sgn}(s) \quad (10)$$

then

$$s \dot{s} = \lambda_n s^2 - |s| (1 + \mathbf{c}^T \mathbf{N}\mathbf{x}) \left\{ 1 - \text{sgn}(s) \frac{\mathbf{k}^T \mathbf{x}}{1 + \mathbf{c}^T \mathbf{N}\mathbf{x}} \right\}. \quad (11)$$

It is evident that if

$$1 + \mathbf{c}^T \mathbf{N}\mathbf{x} > \left| \mathbf{k}^T \mathbf{x} \right| \quad (12)$$

then $s \dot{s} < 0$ for $s \neq 0$, i.e., the reaching and sliding condition is guaranteed [7]. In other words, the system satisfying (12) is able to reach and slide along the sliding mode. For convenience, we call (12) the reaching-and-sliding region or RAS-region in brief. It is equivalent to $1 + \mathbf{c}^T \mathbf{N}\mathbf{x} > \mathbf{k}^T \mathbf{x} > -(1 + \mathbf{c}^T \mathbf{N}\mathbf{x})$ or

$$\text{RAS-region:} \quad \begin{cases} 1 + (\mathbf{c}^T \mathbf{N} - \mathbf{k}^T) \mathbf{x} > 0 \\ 1 + (\mathbf{c}^T \mathbf{N} + \mathbf{k}^T) \mathbf{x} > 0. \end{cases} \quad (13)$$

Intuitively, there exists a subregion $\|\mathbf{x}\| \leq \rho$ bounded by (13). The maximal value of ρ is defined as the RAS-index, expressed by

$$\rho_1 = \min \left\{ \frac{1}{\|\mathbf{c}^T \mathbf{N} + \mathbf{k}^T\|}, \frac{1}{\|\mathbf{c}^T \mathbf{N} - \mathbf{k}^T\|} \right\}. \quad (14)$$

A larger ρ_1 implies a larger RAS-region. Later it will be found that a larger ρ_1 is accompanied with a slower convergence rate to the origin. Clearly, the design of sliding function is a compromise between the RAS-index and the system convergence rate.

Actually, the RAS-region only ensures the reaching and sliding behavior. For the system stability, a subregion in (13) subject to $s = 0$ should be further decided, which is called the stable-sliding region or SS-region in brief. To effectively describe the sliding motion $s = 0$, let us define

$$\begin{bmatrix} \mathbf{y}_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda}_{n-1} & \mathbf{0} \\ \mathbf{0}^T & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{V}_{n-1}^T \\ \mathbf{v}_n^T \end{bmatrix} \mathbf{x}. \quad (15)$$

From (6) and (7), if $s = 0$ then $\mathbf{v}_n^T \mathbf{x} = 0$, i.e., $y_n = 0$. Therefore, rearranging (15) yields

$$\begin{aligned} \mathbf{x} &= [\mathbf{W}_{n-1} \ \mathbf{w}_n] \begin{bmatrix} \mathbf{\Lambda}_{n-1}^{-1} & \mathbf{0} \\ \mathbf{0}^T & \lambda_n^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{n-1} \\ y_n \end{bmatrix} \\ &= \mathbf{W}_{n-1} \mathbf{\Lambda}_{n-1}^{-1} \mathbf{y}_{n-1}. \end{aligned} \quad (16)$$

Clearly, $\mathbf{y}_{n-1} = \mathbf{0}$ leads to $\mathbf{x} = \mathbf{0}$ for $s = 0$. A candidate of Lyapunov function is then given as for $s=0$

$$L = \frac{1}{2} \mathbf{y}_{n-1}^T \mathbf{y}_{n-1} \quad \text{for } s = 0. \quad (17)$$

From (2), (4), and (15), its derivative with respect to time becomes

$$\begin{aligned}\dot{L} &= \mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{V}_{n-1}^T (\mathbf{A} \mathbf{x} + \mathbf{b} u_{\text{eq}} + \mathbf{N} \mathbf{x} u_{\text{eq}}) \\ &= \mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{V}_{n-1}^T ((\mathbf{A} - \mathbf{b} \mathbf{k}^T) \mathbf{x} + \mathbf{b} \mathbf{k}^T \mathbf{x} + \mathbf{b} u_{\text{eq}} + \mathbf{N} \mathbf{x} u_{\text{eq}}) \\ &= \mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{y}_{n-1} + \mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{V}_{n-1}^T (\mathbf{b} \mathbf{k}_x^T + \mathbf{b} u_{\text{eq}} + \mathbf{N} \mathbf{x} u_{\text{eq}}).\end{aligned}\quad (18)$$

Note that based on the concept of equivalent control [7], the bang–bang input u has been replaced by the continuous equivalent control u_{eq} , expressed by

$$u_{\text{eq}} = -\frac{\mathbf{k}^T \mathbf{x}}{1 + \mathbf{c}^T \mathbf{N} \mathbf{x}} \quad (19)$$

which is derived from (9) by letting $s = 0$ and $\dot{s} = 0$. From (16) and (19), we have

$$\mathbf{k}^T \mathbf{x} = -\left(1 + \mathbf{c}^T \mathbf{N} \mathbf{W}_{n-1} \mathbf{\Lambda}_{n-1}^{-1} \mathbf{y}_{n-1}\right) u_{\text{eq}} \quad (20)$$

and then (18) is changed into

$$\dot{L} = \mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{y}_{n-1} - \mathbf{y}_{n-1}^T \mathbf{H} \mathbf{y}_{n-1} \frac{\mathbf{k}^T \mathbf{x}}{1 + \mathbf{c}^T \mathbf{N} \mathbf{x}} \quad (21)$$

where

$$\mathbf{H} = \mathbf{\Lambda}_{n-1} \mathbf{V}_{n-1}^T (\mathbf{I}_n - \mathbf{b} \mathbf{c}^T) \mathbf{N} \mathbf{W}_{n-1} \mathbf{\Lambda}_{n-1}^{-1}.$$

If $\dot{L} < 0$, then (17) is really a Lyapunov function, which means the system in $s = 0$ performs a stable sliding motion converging to the origin. Hence, the SS-region is defined as

SS-region:

$$(1 + \mathbf{c}^T \mathbf{N} \mathbf{x}) \mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{y}_{n-1} - (\mathbf{k}^T \mathbf{x}) \mathbf{y}_{n-1}^T \mathbf{H} \mathbf{y}_{n-1} < 0 \quad (22)$$

where the truth of $1 + \mathbf{c}^T \mathbf{N} \mathbf{x} > 0$ is adopted. Interestingly, if $\mathbf{N} = \mathbf{b} \mathbf{d}^T$ then $\mathbf{H} = \mathbf{0}$ for arbitrary \mathbf{d} . In this case, the SS-region is not needed to be considered since from C1) $\mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{y}_{n-1} < 0$. Only the RAS-region (13) is required.

For a second-order bilinear system, (22) is reduced to

$$\text{SS-region for } n = 2: \quad 1 + \left(\mathbf{c}^T \mathbf{N} - \frac{h}{\lambda_1} \mathbf{k}^T\right) \mathbf{x} > 0 \quad (23)$$

where $h = \mathbf{v}_1^T (\mathbf{I}_2 - \mathbf{b} \mathbf{c}^T) \mathbf{N} \mathbf{w}_1$. While for a higher order system, the expression becomes

$$\begin{aligned}& (1 + \mathbf{c}^T \mathbf{N} \mathbf{x}) \mathbf{y}_{n-1}^T \mathbf{\Lambda}_{n-1} \mathbf{y}_{n-1} - (\mathbf{k}^T \mathbf{x}) \mathbf{y}_{n-1}^T \mathbf{H} \mathbf{y}_{n-1} \\ & < -\left(|\lambda|_{\min} (1 + \mathbf{c}^T \mathbf{N} \mathbf{x}) - |\mathbf{k}^T \mathbf{x}| \|\mathbf{H}\|\right) \|\mathbf{y}_{n-1}\|^2\end{aligned}\quad (24)$$

where $|\lambda|_{\min} = \min\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_{n-1}|\}$. Obviously, if $|\lambda|_{\min} (1 + \mathbf{c}^T \mathbf{N} \mathbf{x}) > |\mathbf{k}^T \mathbf{x}| \|\mathbf{H}\|$ then $\dot{L} < 0$. Thus, the SS-region can be further restricted to $|\lambda|_{\min} (1 + \mathbf{c}^T \mathbf{N} \mathbf{x}) > |\mathbf{k}^T \mathbf{x}| \|\mathbf{H}\|$. It is equivalent to

$$\text{SS-region for } n \geq 3: \quad \begin{cases} 1 + \left(\mathbf{c}^T \mathbf{N} - \frac{\|\mathbf{H}\|}{|\lambda|_{\min}} \mathbf{k}^T\right) \mathbf{x} > 0 \\ 1 + \left(\mathbf{c}^T \mathbf{N} + \frac{\|\mathbf{H}\|}{|\lambda|_{\min}} \mathbf{k}^T\right) \mathbf{x} > 0. \end{cases} \quad (25)$$

Significantly, the RAS-region (13) and the SS-region (23) for $n = 2$ and (25) for $n \geq 3$ are all in the form of linear inequalities. This indeed simplifies the stability analysis. Most significantly, the neighborhood

near the origin satisfies all these inequalities; in other words, the origin is an asymptotically stable equilibrium point. Before getting into the numeric examples, a property concerning the convergence rate to the origin during $s = 0$ should be emphasized here. From (17), (21), and (24), we have

$$\frac{d}{dt} \|\mathbf{y}_{n-1}\|^2 < -2 \left(|\lambda|_{\min} - \frac{|\mathbf{k}^T \mathbf{x}|}{1 + \mathbf{c}^T \mathbf{N} \mathbf{x}} \|\mathbf{H}\| \right) \|\mathbf{y}_{n-1}\|^2. \quad (26)$$

Clearly, if $s = 0$ then $|\lambda|_{\min} = \min\{|\lambda_1|, |\lambda_2|, \dots, |\lambda_{n-1}|\}$ is related to the convergence rate of $\|\mathbf{y}_{n-1}\|^2$, or the convergence rate to the origin $\mathbf{x} = \mathbf{0}$. It is easy to conclude that the larger $|\lambda|_{\min}$, the faster the convergence rate.

Besides, one limitation related to the system initial condition should be addressed here: The initial system state $\mathbf{x}(0)$ has to be located within the RAS-region. In practice, it is difficult to locate the initial system state $\mathbf{x}(0)$ precisely. In case that $\mathbf{x}(0)$ is not in the RAS-region, it is required to use other control algorithms to drive the system into the RAS-region first. For example, a conventional linear state-feedback control with high gain may be adopted to control a system of (1) possessing small \mathbf{N}_k . In the next section, to appropriately demonstrate the bang–bang sliding control, the system state is assumed to be initially located in the RAS-region, as mentioned.

III. NUMERIC EXAMPLES OF SECOND-ORDER BILINEAR SYSTEMS

Consider the following second-order bilinear systems, governed by:

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{b} u + \mathbf{N} \mathbf{x} u \quad (27)$$

where $\mathbf{x} = [x_1 \ x_2]^T$, $u \in \{1, -1\}$ and the system matrices

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}.$$

There are two numeric examples discussed. The first one adopts the work of Longchamp [12] to control the continuously stirred tank reactor with a first-order irreversible exothermic reaction under nonisothermal conditions. In this example, the condition $\mathbf{N} = \mathbf{b} \mathbf{d}^T$ is satisfied, so only the RAS-region is required. Further, in order to demonstrate the case of $\mathbf{N} \neq \mathbf{b} \mathbf{d}^T$, the second example is introduced by directly modifying the \mathbf{N} matrix in the first example.

Based on the conditions C1)–C3) the gain vector $\mathbf{k}^T = [k_1 \ k_2]$ is obtained by assigning eigenvalues λ_1 and λ_2 for $\mathbf{A} - \mathbf{b} \mathbf{k}^T$. Note that λ_1 is related to the convergence rate to the origin during the sliding mode $s = 0$, as shown in (26). The sliding function is then determined by $s = \mathbf{c}^T \mathbf{x}$ with $\mathbf{c}^T = \mathbf{k}^T (\mathbf{A} - \lambda_2 \mathbf{I}_2)^{-1} = [c_1 \ c_2]$. Now, the RAS-region (13) becomes

$$\begin{aligned}1 + (c_1 n_{11} c_2 n_{21} - k_1) x_1 + (c_1 n_{12} c_2 n_{22} - k_2) x_2 &> 0 \\ 1 + (c_1 n_{11} c_2 n_{21} + k_1) x_1 + (c_1 n_{12} c_2 n_{22} + k_2) x_2 &> 0\end{aligned}\quad (28)$$

and the SS-region (23) is rewritten as

$$\begin{aligned}1 + \left(c_1 n_{11} + c_2 n_{21} - \frac{h k_1}{\lambda_1}\right) x_1 \\ + \left(c_1 n_{12} + c_2 n_{22} - \frac{h k_2}{\lambda_1}\right) x_2 > 0\end{aligned}\quad (29)$$

where $h = \mathbf{v}_1^T (\mathbf{I}_2 - \mathbf{b} \mathbf{c}^T) \mathbf{N} \mathbf{w}_1$. Here, \mathbf{v}_1^T and \mathbf{w}_1 are the left- and right eigenvectors corresponding to λ_1 . Note that the RAS-region (28) applies to both examples, while the SS-region (29) is only used for the second example because $\mathbf{N} \neq \mathbf{b} \mathbf{d}^T$. Finally, as mentioned before, the origin $\mathbf{x} = \mathbf{0}$ is asymptotically stable.

TABLE I
RAS-INDEX OF EXAMPLE 1

λ_1	-0.1	-0.5	-1.0	-1.5	-2.0	-2.5	-3.0
λ_2	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7
ρ_1	0.0611	0.0555	0.0499	0.0453	0.0414	0.0382	0.0354

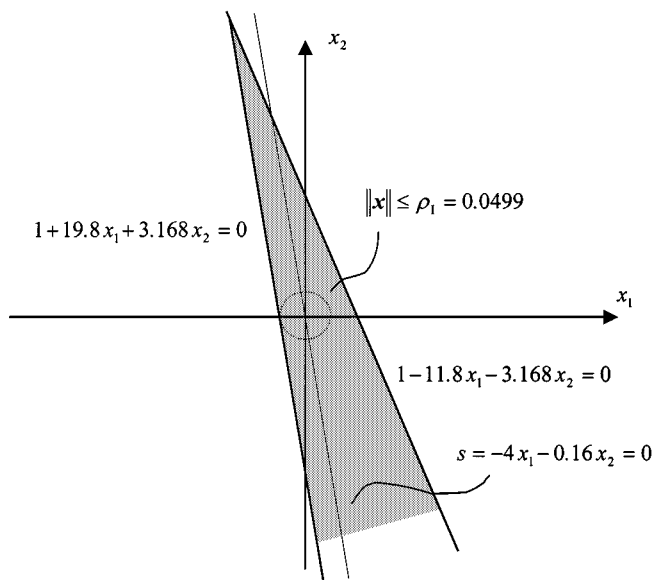


Fig. 1. RAS-region (shaded) of Example 1.

Example-1: Consider the continuously stirred tank reactor with a first-order irreversible exothermic reaction under nonisothermal conditions, expressed as (27) with

$$\mathbf{A} = \begin{bmatrix} 4.25 & 1 \\ -6.25 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -0.25 \\ 0 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \quad (30)$$

The spectrum of \mathbf{A} is $\{3, -0.75\}$ and $\mathbf{N} = \mathbf{b}\mathbf{d}^T$ with $\mathbf{d}^T = [4 \ 0]$. Hence, only the RAS-region (27) is needed for the stability analysis. Let $\lambda_2 = -0.7$, which is not in the spectrum of \mathbf{A} , and the convergence rate λ_1 range from -0.1 to -3.0 . Table I lists the RAS-index ρ_1 and shows that a faster convergence rate results in a smaller ρ_1 . The case of $\lambda_1 = -1.0$ and $\lambda_2 = -0.7$ are selected for demonstration and then the feedback gain is obtained as $\mathbf{k}^T = [-15.8 \ -3.168]$. Therefore, the sliding function in (7) is set as $s = -4x_1 - 0.64x_2$. Fig. 1 depicts the RAS-region in the phase plane with $\rho_1 = 0.0499$. Let the system be initially located at $\mathbf{x}^T(0) = [0.05 \ 0.05]$ in the RAS-region. Then the system trajectory shown in Fig. 2 is driven to $s = 0$ and performs the sliding motion. The state variables x_1 and x_2 are given in Fig. 3. After reaching the sliding mode at time $t_r = 0.31$ s, these two state variables converge to the origin $\mathbf{x} = 0$ with a convergence rate $\lambda_1 = -1.0$, not affected by \mathbf{N} . It can be seen from (26) with $\mathbf{H} = 0$ since $\mathbf{N} = \mathbf{b}\mathbf{d}^T$. Obviously, the origin is asymptotically stable.

For comparison, the stability region in the work of Longchamp [12] is found about $\|\mathbf{x}\| \leq 0.045$ in the similar sense of RAS-index. It is approximate to the region $\|\mathbf{x}\| \leq \rho_1$ as given in Table I. However, the stability analysis introduced in this paper is simpler due to the fact that only linear inequalities are processed. \square

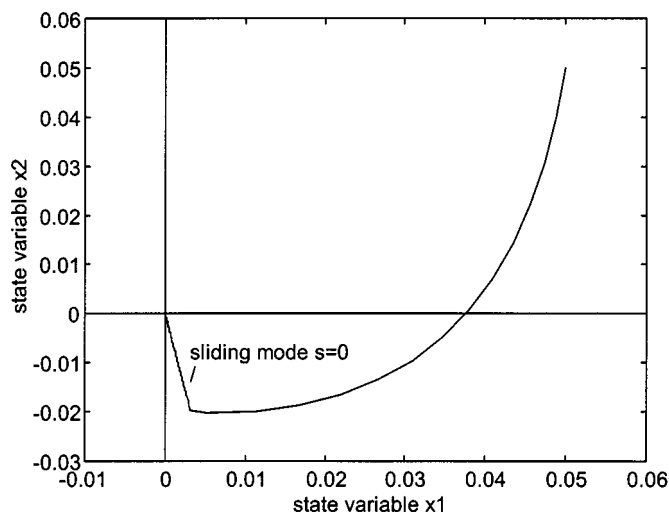


Fig. 2. The system trajectory in the phase plane of Example 1.

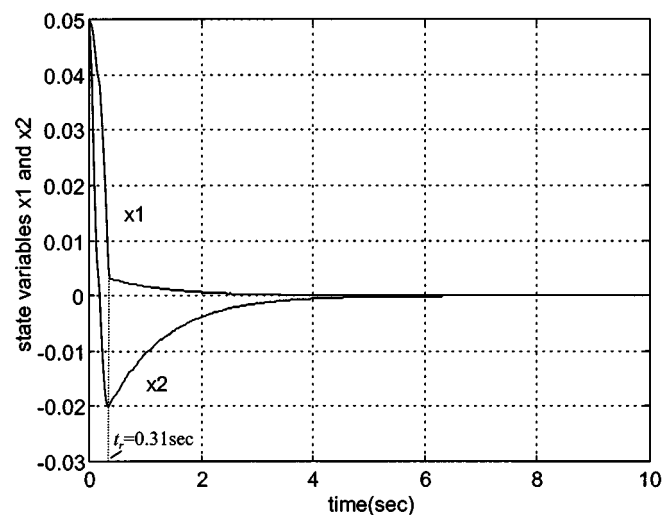


Fig. 3. The state variables of Example 1.

TABLE II
RAS-INDEX OF EXAMPLE 2

λ_1	-0.1	-0.5	-1.0	-1.5	-2.0	-2.5	-3.0
λ_2	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7	-0.7
ρ_1	0.0608	0.0558	0.0504	0.0458	0.0420	0.0387	0.0359

Example 2: To demonstrate the bang–bang sliding controller is still useful for the case of $\mathbf{N} \neq \mathbf{b}\mathbf{d}^T$, this example employs the same model in (30) except that

$$\mathbf{N} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Clearly, $\mathbf{N} \neq \mathbf{b}\mathbf{d}^T$ and the SS-region (29) should be included for the stability analysis. The eigenvalue λ_2 is also set as -0.7 , not in the spectrum of \mathbf{A} , and the eigenvalue λ_1 ranges from -0.1 to -3.0 . Table II lists the RAS-index and shows that a faster convergence rate results in a smaller RAS-index. Let $\lambda_1 = -1.0$ and $\lambda_2 = -0.7$, then $\mathbf{k}^T = [-15.8 \ -3.168]$ and $s = -4x_1 - 0.64x_2$, same as those in the Example 1. Fig. 4 illustrates the RAS-region and the SS-region in the phase plane. Figs. 5 and 6 are obtained by setting the initial condition

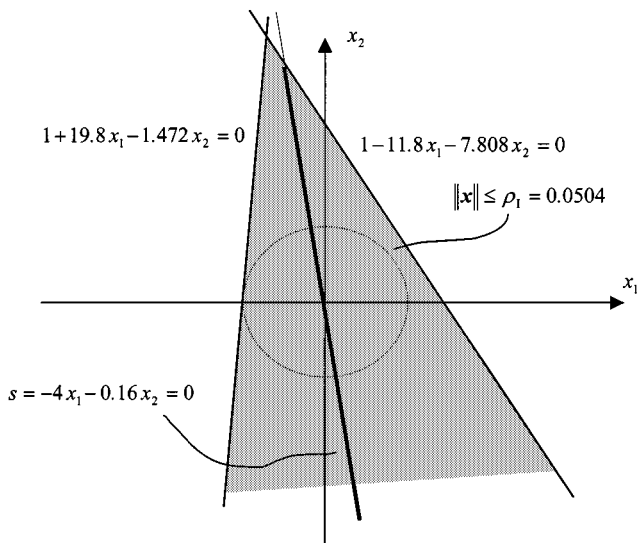


Fig. 4. RAS-region (shaded) and SS-region (bold line) of example 2.

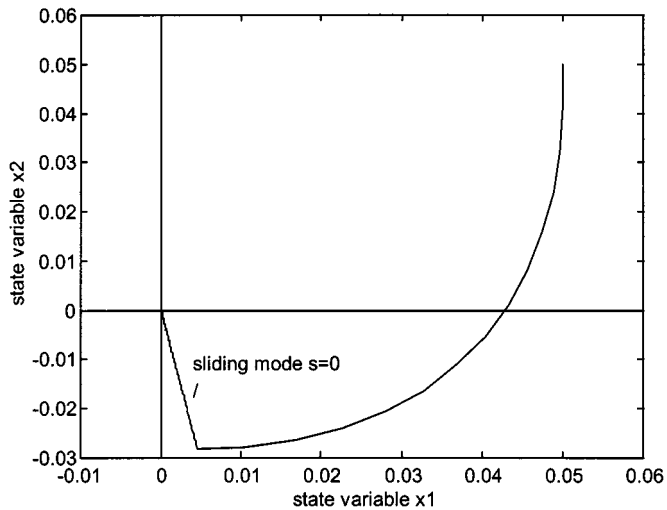


Fig. 5. The system trajectory in the phase plane of example 2.

as $\mathbf{x}^T(0) = [0.05 \ 0.05]$, which is located in the RAS-region. From Fig. 5, the system trajectory is first steered to $s = 0$ and then restricted in the sliding mode. Fig. 6 shows the success of controlling x_1 and x_2 to the origin $\mathbf{x} = 0$. The state variables x_1 and x_2 are given in Fig. 6. After reaching the sliding mode at time $t_r = 0.32$ s, these two state variables converge to the origin $\mathbf{x} = 0$ with convergence rates affected by $\lambda_1 = -1.0$ and $\mathbf{H} \neq 0$ since $\mathbf{N} \neq \mathbf{b}\mathbf{d}^T$. It is evident the origin is asymptotically stable. \square

Note that the success of the bang–bang sliding controller is demonstrated from the above simulation results. Most importantly, it is quite straightforward to design the sliding function by the pole-assignment-based method. Besides, it is simple to analyze the system stability from the RAS-region and the SS-region, each expressed by a set of linear inequalities.

IV. CONCLUSIONS

A new design approach for bang–bang sliding control of a class of single-input bilinear systems is introduced. The sliding function is determined by the pole-assignment-based method. Three conditions are required to choose the eigenvalues, which are related to the system

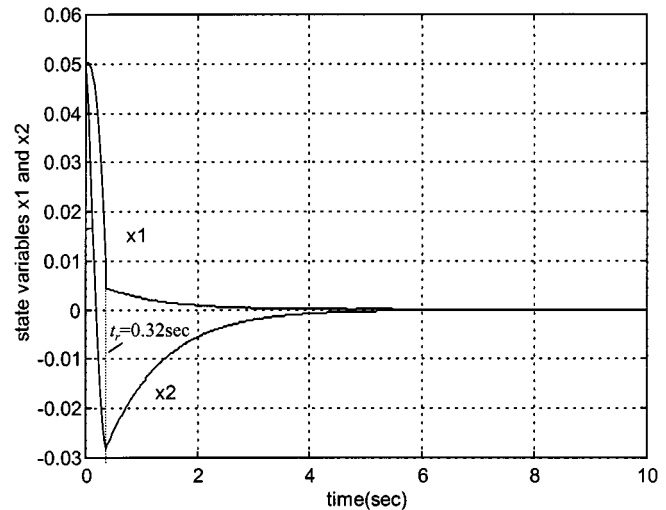


Fig. 6. The state variables of example 2.

stability and the convergence rate in the sliding mode. The origin is found to be asymptotically stable. The stability region depends on the RAS-region and the SS-region, each expressed by a set of linear inequalities. Besides, an RAS-index is produced to show whether the RAS-region is reasonable or not. It is noticed that the sliding function design is a compromise of the RAS-index and the convergence rate to the origin. Finally, simulation results of two second-order bilinear systems are adopted for demonstration. Investigations to extend the approach to multi-input bilinear systems are in progress.

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