Optimal 1-edge fault-tolerant designs for ladders ✤

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Abstract
A graph $G^*$ is 1-edge fault-tolerant with respect to a graph $G$, denoted by 1-EFT($G$), if every graph obtained by removing any edge from $G^*$ contains $G$. A 1-EFT($G$) graph is optimal if it contains the minimum number of edges among all 1-EFT($G$) graphs. The $k$th ladder graph, $L_k$, is defined to be the cartesian product of the $P_k$ and $P_2$ where $P_n$ is the $n$-vertex path graph.

In this paper, we present several 1-edge fault-tolerant graphs with respect to ladders. Some of these graphs are proven to be optimal.

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1. Introduction and notations
In this paper, any graph means an undirected graph in which multiple edges are allowed. Let $G = (V, E)$ be a graph where $V = V(G)$ is the vertex set of $V$, $deg_G(x)$ denotes its degree in $G$. Let $E'$ be a subset of $E$. We use $G - E'$ to denote the spanning subgraph of $G$ with its edge set to be $E - E'$. For convenience, $G - e$ denotes $G - \{e\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The cartesian product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is the smallest graph with the vertex set $V_1 \times V_2$ such that the subgraph induced by $V_1 \times \{v_2\}$ is isomorphic to $G_1$ for every $v_2 \in V_2$, and the subgraph induced by $\{v_1\} \times V_2$ is isomorphic to $G_2$ for every $v_1 \in V_1$.

Motivated by the study of computers and communication networks that tolerate failure of their components, Harary and Hayes [6] have formulated the concept of edge fault tolerance in graphs. Given a target graph $G = (V, E)$, let $G^* = (V, E^*)$ be a spanning supergraph of $G$. $G^*$ is said to be k-EFT($G$), if $G^* - F$ contains a subgraph isomorphic to $G$, which is called a reconfiguration for $k$-edge fault $F$ (or simply reconfiguration), for any $F \subset E^*$ and $|F| = k$. A reconfiguration can be viewed as a relabeling of vertices of $G^*$ such that $G^* - F$ contains $G$. We sometimes write “$G^*$ is a k-EFT($G$) graph” as “$G^*$ is a k-EFT($G$)”, for short. The graph $G^*$ is said to be optimal if $G^*$ contains the smallest number of edges among all k-EFT($G$) graphs. We use $eft_k(G)$ to denote the difference between the number of edges in an opti-
mal $k$-EFT($G$) graph and that in $G$. Families of $k$-EFT graphs with respect to some graphs have been studied in literature [1,2,4,6–8,10–17].

The $n$-dimensional mesh $M(m_1, m_2, \ldots, m_n)$ is defined to be the cartesian product $P_{m_1} \times P_{m_2} \times \cdots \times P_{m_n}$ of $n$ paths. Mesh is a widely used graph model for computer networks [9]. Farrag [4] has presented families of 1-EFT graphs with respect to the $n$-dimensional meshes. In [6], the graph $C(m_1, m_2, \ldots, m_n) = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_n}$ was proposed as a 1-EFT graphs with respect to the $n$-dimensional mesh $M(m_1, m_2, \ldots, m_n)$. We call such graphs multidimensional torus graphs because their construction is similar to that of the torus for $n = 2$ [5]. Harary and Hayes [6] conjectured that these multidimensional torus graphs are optimal if $m_i \geq 3$ for every $i$. There is another 1-EFT graph for the $n$-dimensional meshes. We assume the vertices of $M(m_1, m_2, \ldots, m_n)$ are labeled canonically. Thus, $x_{i_1, i_2, \ldots, i_n}$ is a vertex of $M(m_1, m_2, \ldots, m_n)$ if and only if $1 \leq i_j \leq m_j$ for $1 \leq j \leq n$. Moreover, $x_{i_1, i_2, \ldots, i_n}$ is adjacent to another vertex $x_{j_1, j_2, \ldots, j_n}$ if there exist a index $k$ such that $|i_k - j_k| = 1$ and $i_l = j_l$ for all indices $l \neq k$. Then, $V_p = \{x_{i_1, i_2, \ldots, i_n} | i_k = 1 \text{ or } m_k \text{ for some } 1 \leq k \leq n \}$ is the set of peripheral vertices. Let $L_{k1}, L_{k2}, \ldots, L_{kn}$ be a vertex in $V_p$. The antipodal vertex of $x_{i_1, i_2, \ldots, i_n}$ is $x_{j_1, j_2, \ldots, j_n}$, with $j_k = m_k - i_k + 1$, which is another vertex in $V_p$. It is easy to check that every vertex in $V_p$ has exactly one antipodal. In $M(m_1, m_2, \ldots, m_n)$, we add the edges joining each vertex in $V_p$ to its antipodal counterpart to form a new graph $P(m_1, m_2, \ldots, m_n)$. We call these $P(m_1, m_2, \ldots, m_n)$ projective-plane graphs because their construction is similar to that of the projective plane when $n = 2$ [5]. It is proven in [3] that $P(m_1, m_2, \ldots, m_n)$ is also 1-EFT($M(m_1, m_2, \ldots, m_n)$) and it contains fewer edges than that of $C(m_1, m_2, \ldots, m_n)$. Thus, the conjecture posed in [6] is disproved with these projective-plane graphs.

The projective-plane graphs are optimal for some cases but not for all. Note that every $n$-dimensional hypercube can be viewed as the mesh $M(2, 2, \ldots, 2)$. Our $P(2, 2, \ldots, 2)$ is actually the same 1-EFT graph as that proposed in [1,6,7,13,16]. Thus, $P(2, 2, \ldots, 2)$ is an optimal 1-EFT graph. It is proved in [3] that the graph in Fig. 1 (a) is a 1-EFT($M(3, 2)$) and the graph in Fig. 1 (b) is a 1-EFT($M(4, 2)$). With these two examples, we know that the projective-plane graphs may not be optimal for some cases. Furthermore, the problem of finding the optimal 1-EFT for all $n$-dimensional meshes remains unsolved.

In this paper, we only aim at the 1-EFT graphs for $M(k, 2)$ with $k \geq 2$. For simplicity, the $k$th ladder graph $L_k$ is defined to be $M(k, 2)$. Since the projective-plane graph $P(k, 2)$ is a 1-EFT($L_k$) graph, we know that eft($L_k$) $\leq k$. In this paper, we will prove by constructing a 1-EFT($L_k$) graph $L^*_k$ that eft($L_k$) $\leq k - 1$ if $k$ is odd and $k \geq 7$, and eft($L_k$) $\leq k - 2$ if $k$ is even and $k \geq 4$. Moreover, we prove that eft($L_2$) = eft($L_3$) = eft($L_4$) = 2, and eft($L_5$) = 3.

2. Some 1-EFT designs for ladders

The vertices of $L_k$ can be labeled by $x_{i,j}$ with $1 \leq i \leq k$ and $1 \leq j \leq 2$ canonically. The vertices $x_{1,1}, x_{k,1}, x_{1,2}$, and $x_{k,2}$ are called the corner vertices of $L_k$.

We have the following theorem:

**Theorem 1.** Let $L^*_k$ be a 1-EFT($L_k$) graph. Then we have

(i) $\deg_{L^*_k}(x) \geq 3$ for any vertex $x$ of $L^*_k$, and
(ii) $\eft(L_k) \geq 2$.

**Proof.** Suppose some vertex $x$ with $\deg_{L^*_k}(x) = 2$. Let $e$ be any edge incident with $x$. Obviously, $\deg_{L^*_k-e}(x) = 1$. Since $\deg_{L_k-e}(x) \geq 2$ for any vertex $x$ of $L_k$, $L_k$ is not a subgraph of $L^*_k - e$. We obtain a contradiction that $L^*_k$ is a 1-EFT($L_k$) graph. Hence, $\deg_{L^*_k}(x) \geq 3$. Since there are exactly four corner vertices in every $L_k$, we have $\eft(L_k) \geq 2$. □

**Corollary 1.** $\eft(L_k) \geq 2$ if $k > 4$.

**Proof.** It is observed that there are exactly three different ways of joining the four corner vertices in $L_k$ with two edges, namely $\{x_{1,1}, x_{1,2}\}, \{x_{1,1}, x_{k,2}\}, \{x_{1,1}, x_{k,1}\}$.

![Fig. 1. (a) A 1-EFT($M(3, 2)$), $L^*_3$; (b) a 1-EFT($M(4, 2)$), $L^*_4$.](image-url)
Let $L^*_k$ ($L^*_2$ and $L^*_4$, respectively) be the graph $P(2, 2)$ (the graph in Figs. 1(a) and 1(b), respectively).

From the above discussion, $L^*_k$ is 1-EFT$(L_k)$ for $k = 2, 3, \text{ and } 4$. Since there are exactly 2 edges added to $L_k$ with $k = 2, 3, \text{ and } 4$, by Theorem 1 these graphs are optimal. It can be verified that the optimal 1-EFT$(L_k)$ is unique for $k = 2, 3, \text{ and } 4$ by checking all the three cases joining two edges to the corner vertices of $L_k$. We obtain the following theorem:

**Theorem 2.** $\text{eft}_1(L_k) = 2$ for $k = 2, 3, \text{ and } 4$.

2.2. **An optimal 1-EFT$(L_5)$ graph**

Consider the spanning supergraph $L^*_5$ of $L_5$ given by $E(L^*_5) = E(L_5) \cup \{(x_{1,1}, x_{5,2}), (x_{1,2}, x_{4,2}), (x_{2,1}, x_{5,1})\}$ as shown in Fig. 2.

Edges of $L_5$ can be divided into the following 7 classes: namely,

- $A = \{(x_{1,1}, x_{1,2})\}$,
- $B = \{(x_{i,1}, x_{i,2}) \mid 2 \leq i \leq 4\}$,
- $C = \{(x_{5,1}, x_{5,2})\}$,
- $D = \{(x_{1,1}, x_{2,1}), (x_{1,2}, x_{2,2})\}$,
- $E = \{(x_{2,1}, x_{3,1}), (x_{2,2}, x_{3,2})\}$,
- $F = \{(x_{3,1}, x_{4,1}), (x_{3,2}, x_{4,2})\}$,
- $G = \{(x_{4,1}, x_{5,1}), (x_{4,2}, x_{5,2})\}$.

We can reconfigure $L_5$ in $L^*_5$ for any faulty edge $e$ in $A, B, C, D, E, F, \text{ and } G$, respectively) as shown in Figs. 3(a), 3(b), 3(c), 3(d), 3(e), 3(f), and 3(g), respectively). Hence $L^*_5$ is 1-EFT$(L_5)$. The following theorem follows from Corollary 1.

**Theorem 3.** $\text{eft}_1(L_5) = 3$.

2.3. 1-EFT$(L_k)$ for graphs where $k > 4$ and even

In this subsection, we are going to construct 1-EFT$(L_k)$ graphs where $k$ is an even integer with $k \geq 4$.

Let the spanning supergraph $L^*_k$ of $L_k$ be the graph that adds $E' = \{(x_{i,j}, x_{k-i+1,j}) \mid 1 \leq i < k/2, j = 1, 2\}$ to $E(L_k)$ as shown in Fig. 4(a). The graph in Fig. 4(a) is actually isomorphic to $M(k/2, 2, 2)$ as shown in Fig. 4(b). We can reconfigure $L_k$ in $L^*_k$ as shown in Fig. 4(c) for any faulty edge of the form $(x_{i,1}, x_{i,2})$ or as shown in Fig. 4(d) for any faulty edge of the form $(x_{i,1}, x_{i+1,1})$ or $(x_{i,2}, x_{i+1,2})$. Hence, $M(k/2, 2, 2)$ is a 1-EFT$(L_k)$. We obtain the following theorem:

**Theorem 4.** $\text{eft}_1(L_k) \leq k - 2$ where $k$ is an even integer with $k \geq 4$.

![Fig. 2. A 1-EFT($M(5, 2)$), $L^*_5$.](image)

![Fig. 3. A 1-EFT($M(5, 2)$), $L^*_5$.](image)
Fig. 4. (a) $L_k^*$, a 1-EFT($L_k$) where $k$ is even and $k \geq 4$; (b) the 3-dimensional mesh $M(k/2, 2, 2)$; (c) reconfigure $L_k$ for any faulty edge of the form $(x_i, x_{i+1})$; and (d) reconfigure $L_k$ for any faulty edge of the form $(x_i, x_{i+1})$, or $(x_{i+1}, x_{i+2})$.

Fig. 5. A 1-EFT($L_k$) where $k$ is odd and $k \geq 7$.

2.4. 1-EFT($L_k$) graphs for $k \geq 7$ and odd

Assume $k$ is an odd integer with $k \geq 7$. Construct the spanning supergraph $L_k^*$ of $L_k$ by adding $E' = \{(x_{1, i}, x_{4, j}) \mid x_{3, 2}, x_{6, 2}, (x_{2, 1}, x_{5, 3}), x_{4, 4}, x_{7, 1}$, $(x_{1, 1}, x_{5, 2}), (x_{3, 1}, x_{7, 2}) \} \cup \{(x_{2i, j}, x_{2i+1, j}) \mid 3 \leq i \leq (k - 3)/2, j = 1, 2 \}$ as shown in Fig. 5.

Edges of $L_k$ can be divided into the following 7 classes:

$$A = \{(x_{i, 1}, x_{i, 2}) \mid i = 1, 2\} \cup \{(x_{2i, j}, x_{2i+1, j}) \mid 4 \leq i \leq (k - 3)/2, j = 1, 2\};$$

$$B = \{(x_{i, 1}, x_{i, 2}) \mid i = 3, 4\} \cup \{(x_{2i-1, j}, x_{2i, j}) \mid 4 \leq i \leq (k - 1)/2, j = 1, 2\};$$

$$C = \{(x_{5, 1}, x_{5, 2})\} \cup \{(x_{i, 1}, x_{i, 2}) \mid 4 \leq i \leq (k - 1)/2\};$$

$$D = \{(x_{i, 1}, x_{i, 2}) \mid i = 6, 7\} \cup \{(x_{i, 1}, x_{i, 2}) \mid i = k, k - 1\};$$

$$E = \{(x_{1, j}, x_{2, j}) \mid j = 1, 2\} \cup \{(x_{3, j}, x_{4, j}) \mid j = 1, 2\};$$

$$F = \{(x_{2, j}, x_{3, j}) \mid j = 1, 2\} \cup \{(x_{5, j}, x_{6, j}) \mid j = 1, 2\};$$

$$G = \{(x_{4, j}, x_{5, j}) \mid j = 1, 2\} \cup \{(x_{6, j}, x_{7, j}) \mid j = 1, 2\} \cup \{(x_{k-1, j}, x_{k, j}) \mid j = 1, 2\}.$$

We can reconfigure $L_k$ in $L_k^*$ for any faulty edge $e$ in $A$, $B$, $C$, $D$, $E$, $F$, and $G$ respectively as shown in Figs. 6(a), 6(b), 6(c), 6(d), 6(e), 6(f), and 6(g), respectively. Hence $L_k^*$ is 1-EFT($L_k$). We obtain the following theorem:

Theorem 5. $\text{eft}_1(L_k) \leq k - 1$ where $k$ is an odd integer with $k \geq 7$. 
Fig. 6. Reconfigures of $L_k$ in $L^*_k$ where $k$ is odd and $k \geq 7$ for any faulty edge in $A, B, C, D, E, F,$ and $G,$ respectively.

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References


