Brother trees: A family of optimal 1\textsubscript{p}-hamiltonian and 1-edge hamiltonian graphs

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Abstract

In this paper we propose a family of cubic bipartite planar graphs, brother trees, denoted by BT\((n)\) with \(n \geq 2\). Any BT\((n)\) is hamiltonian. It remains hamiltonian if any edge is deleted. Moreover, it remains hamiltonian when a pair of nodes (one from each partite set) is deleted. These properties are optimal. Furthermore, the number of nodes in BT\((n)\) is \(6 \cdot 2^n - 4\) and the diameter is \(2n + 1\).

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1. Introduction

An interconnection network connects the processors of the parallel computer. Its architecture can be represented as a graph in which the nodes correspond to the processors and the edges to the communication links. Hence we use graphs and networks interchangeably. There are many mutually conflicting requirements in designing the topology of computer networks. It is almost impossible to design a network optimum from all aspects. One has to design a suitable network satisfying the requirements. Diameter is one of the major requirements in designing the topology of network. Usually a network with smaller diameter is more preferable. The hamiltonian properties is another requirement. For example, “Token Passing” approach is used in some distributed operation systems. Interconnection network requires the presence of hamiltonian cycles in the structure to meet this approach. Fault tolerance is also desirable in massive parallel systems that have a relatively high probability of failure. A number of fault tolerant designs for specific multiprocessor architectures have been proposed based on graph theoretic models in which the processor-to-processor interconnection structure is represented by a graph.

For the graph definition and notation, we follow [1].

\[ G = (V, E) \text{ is a graph if } V \text{ is a finite set and } E \text{ is a subset of } \{(a, b) \mid (a, b) \text{ is an unordered pair of } V\}. \]

We say that \(V\) is the node set and \(E\) is the edge set of \(G\). Two nodes, \(a\) and \(b\), are adjacent if \((a, b) \in E\). A path is a sequence of consecutive adjacent nodes. A path
is usually delimited by \((x_0, x_1, x_2, \ldots, x_{n-1})\). We use \(P^{-1}\) to denote the path \((x_{n-1}, x_{n-2}, \ldots, x_1, x_0)\) if \(P\) is the path \((x_0, x_1, x_2, \ldots, x_{n-1})\). Let \(d_G(x, y)\) denote the distance between two nodes \(x\) and \(y\) in graph \(G\) and \(D(G)\) denote the diameter of \(G\). A path is called a \textit{hamiltonian path} if its nodes are distinct and span \(V\).

A cycle is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a \textit{hamiltonian cycle} if its nodes are distinct except for the first node and last node and if they span \(V\). A graph is hamiltonian if it contains a hamiltonian cycle. A graph \(G = (V, E)\) is 1-edge hamiltonian if \(G - e\) is hamiltonian for any edge \(e\) in \(E\). Obviously, any 1-edge hamiltonian graph is hamiltonian. A 1-edge hamiltonian graph \(G\) is \textit{optimal} if it contains the least number of edges among all 1-edge hamiltonian graphs with the same number of nodes as \(G\). A graph \(G = (V, E)\) is 1-node hamiltonian if \(G - v\) is hamiltonian for any \(v \in V\). A 1-node hamiltonian graph \(G\) is \textit{optimal} if it contains the least number of edges among all 1-node hamiltonian graphs with the same number of nodes as \(G\). A graph \(G = (V, E)\) is 1-hamiltonian if it contains a hamiltonian path of at least three nodes such that its nodes are distinct and span \(V\).

A graph \(G = (V, E)\) is \textit{1-node hamiltonian} if it is 1-edge hamiltonian and 1-node hamiltonian. A 1-hamiltonian graph \(G\) is \textit{optimal} if it contains the least number of edges among all 1-hamiltonian graphs with the same number of nodes as \(G\). This study of optimal 1-hamiltonian graphs is motivated by optimal fault tolerant token ring design in computer networks. A number of optimal 1-hamiltonian graphs have been proposed [2,5,7]. Obviously, \(\deg_G(x) \geq 3\) for any node \(x\) in a 1-edge hamiltonian, 1-node hamiltonian, or 1-hamiltonian graph \(G\). It has been proven that any 1-hamiltonian regular graph is optimal if and only if it is 3-regular.

Note that any cycle of a bipartite graph contains the same number of nodes in each partite set. Thus, the deletion of a node from a hamiltonian bipartite graph results in a non-hamiltonian graph. However, the fault tolerant hamiltonian property is not the only factor in designing the topology of networks. For example, the hypercube \(Q_n\) is a hamiltonian bipartite graph for \(n \geq 2\). Hence, it is not 1-hamiltonian. When fault occurs, we are interested in the longest cycle in the faulty hypercube [6].

Let \(G\) be a bipartite graph with bipartition \(W\) and \(B\). We use \(\mathcal{F}(G)\) to denote \(\{(c, d) \mid c \in W, d \in B\}\). A hamiltonian bipartite graph is \(1_p\)-\textit{hamiltonian} if \(G - F\) remains hamiltonian for any \(F \in \mathcal{F}(G)\).

A \(1_p\)-hamiltonian graph \(G\) is \textit{optimal} if it contains the least number of edges among all \(1_p\)-hamiltonian graphs with the same number of nodes as \(G\). Obviously, \(\deg_G(x) \geq 3\) for any node \(x\) in a \(1_p\)-hamiltonian graph \(G\). Thus any \(1_p\)-hamiltonian graph that is 3-regular is optimal. In [3], a family of optimal 1-edge hamiltonian and \(1_p\)-hamiltonian bipartite graphs, called \textit{honeycomb rectangular torus}, is discussed. It can be shown that the diameter of the honeycomb rectangular torus \(\Omega(\sqrt{p})\), where \(p\) is the number of nodes.

In this paper, we propose a family of graphs, called \(k\)th brother trees, denoted by \(BT(n)\). The graph \(BT(n)\) is planar, bipartite, 3-regular, 1-edge hamiltonian and \(1_p\)-hamiltonian. The diameter of the brother tree is \(\Theta(\log p)\), where \(p\) is the number of nodes.

2. Definitions and notation

To define brother trees, first we define brother cells. Assume that \(k\) is an integer with \(k \geq 2\). The \(k\)th brother cell \(BC(k)\) is the five tuple \((G_k, w_k, x_k, y_k, z_k)\), where \(G_k = (V, E)\) is a bipartite graph with bipartition \(W\) (white) and \(B\) (black) and contains four distinct nodes \(w_k, x_k, y_k, z_k\). \(w_k\) is the \textit{white terminal}; \(x_k\) the \textit{white root}; \(y_k\) the \textit{black terminal} and \(z_k\) the \textit{black root}.

We can recursively define \(BC(k)\) as follows:

1. \(BC(2)\) is the 5-tuple \((G_2, w_2, x_2, y_2, z_2)\) where \(V(G_2) = \{w_2, x_2, y_2, z_2, s, t\}\) and \(E(G_2) = \{(w_2, s), (s, x_2), (x_2, y_2), (y_2, t), (t, z_2), (w_2, z_2), (s, t)\}\).

2. The \(k\)th brother cell \(BC(k)\) with \(k \geq 3\) is composed of two disjoint copies of \((k - 1)\)th brother cells

\[
BC^1(k - 1) = (G^1_{k-1}, w^1_{k-1}, x^1_{k-1}, y^1_{k-1}, z^1_{k-1}),
\]

\[
BC^2(k - 1) = (G^2_{k-1}, w^2_{k-1}, x^2_{k-1}, y^2_{k-1}, z^2_{k-1}),
\]

a white root \(x_k\), and a black root \(z_k\). To be specific, \(V(G_k) = V(G^1_{k-1}) \cup V(G^2_{k-1}) \cup \{x_k, z_k\}\), \(E(G_k) = E(G^1_{k-1}) \cup E(G^2_{k-1}) \cup \{x_k, z_k\}\),

\[
E(G_k) = E(G^1_{k-1}) \cup E(G^2_{k-1}) \cup \{(x_k, x^1_{k-1}), (z_k, z^1_{k-1}), (x_k, z^2_{k-1}), (y^1_{k-1}, w^2_{k-1}), (y^{2}_{k-1}, w^{1}_{k-1})\},
\]

\(w_k = w^1_{k-1}\), and \(y_k = y^2_{k-1}\)
BC(2), BC(3), and BC(4) are shown in Fig. 1. We note that BC\(^1\)(k - 1) and BC\(^2\)(k - 1) are isomorphic for k \(\geq\) 3. This property is referred to as the symmetrical property of BC(k). For this reason, we define the degenerate case, BC(1), as the 5-tuple \((G_1, w_1, x_1, y_1, z_1)\) as \(V(G_1) = \{w_1, y_1\}, E(G_1) = [(w_1, y_1)]\) such that \(x_1 = w_1\) and \(y_1 = z_1\).

We can also define the brother cell BC(k) from the complete binary tree B(k), where \(V(B(k)) = \{1, 2, \ldots, 2^k - 1\}\) and \(E(B(k)) = \{(i, j) \mid \lfloor j/2 \rfloor = i\}\). Assume that k is a positive integer with \(k \geq 2\). The kth brother cell BC\(^k\)(k) = \((G_k, w_k, x_k, y_k, z_k)\) can be constructed by combining two B\(^k\)'s, the upper tree B\(^k\)_\(u\) and the lower tree B\(^k\)_\(l\), and adding edges between their leaf nodes.

Let n be a positive integer with \(n \geq 1\). The brother tree, BT\((n)\), is composed of an \((n + 1)\)th brother cell BC\((n + 1)\) = \((G_{n+1}^1, w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1})\) and an nth brother cell BC\((n)\) = \((G_n^2, w_n, x_n, y_n, z_n)\) with \(V(G_{n+1}^1) \cap V(G_n^2) = \emptyset\). To be specific, \(V(BT(n)) = V(G_{n+1}^1) \cup V(G_n^2)\) and \(E(BT(n)) = E(G_{n+1}^1) \cup E(G_n^2)\) ∪ \([(z_{n+1}^1, x_{n+1}^1), (y_{n+1}^1, w_{n+1}^1), (x_{n+1}^1, z_{n+1}^1), (w_{n+1}^1, y_{n+1}^1)]\).

BT(1) and BT(3) are shown in Fig. 2. Obviously, BT\((n)\) is a 3-regular bipartite planar graph with 6 \(\cdot\) \(2^n\) - 4 nodes. Because the \((n + 1)\)th brother cell is composed of two disjoint nth brother cells and two terminals, the nth brother tree BT\((n)\) is composed of three disjoint nth brother cells, BC\(^1\)(n), BC\(^2\)(n), BC\(^3\)(n) and two terminals, \([x_{n+1}^1, z_{n+1}^1]\). Moreover, BC\(^1\)(n), BC\(^2\)(n) and BC\(^3\)(n) are arranged in a cyclic order in BT\((n)\). Thus any two nodes of BT\((n)\) are in the union of two in the node set of BC\(^1\)(n), BC\(^2\)(n), BC\(^3\)(n) and \([x_{n+1}^1, z_{n+1}^1]\). For this reason, we can assume without loss of generality that any two nodes of BT\((n)\) are in G\(^1\)\(_{n+1}\) and any edge of BT\((n)\) is in G\(^1\)\(_{n+1}\). This property is referred to as the symmetrical property of BT\((n)\).

### 3. Diameter

**Theorem 3.1.** \(D(BT(n)) = 2n + 1\) for any positive integer \(n \geq 1\).
Proof. It is easy to prove by induction that
\[ d_{BT(n)}(x_{n+1}^1, z_{n+1}^1) = 2n + 1. \]
Let \( u \) and \( v \) be any two nodes of \( BT(n) \). We will prove that
\[ d_{BT(n)}(u, v) \leq 2n + 1. \]
Using the symmetrical property of brother trees, we may assume that \( u \) and \( v \) are in the brother cell \( G_{n+1}^1 \). Note that the brother cell \( G_{n+1}^1 \) is composed of two complete binary trees \( B(n+1)_u \) and \( B(n+1)_l \). Thus,
\begin{enumerate}
\item[(1)] both \( u \) and \( v \) are in \( V(B(n+1)_u) \), or both \( u \) and \( v \) are in \( V(B(n+1)_l) \), or
\item[(2)] \( u \in V(B(n+1)_u) \) and \( v \in V(B(n+1)_l) \).
\end{enumerate}

Now, we introduce some notations before our proof. Let \( V(B(n+1)_u) = \{1, 2, \ldots, 2^{2n+1} - 1\} \) and \( V(B(n+1)_l) = \{1', 2', \ldots, (2^{2n+1} - 1)\} \). Now, join \( B(n+1)_u \) and \( B(n+1)_l \) with the edge set \( \{(2^n + i, 2^n + i'), (2^n + i, 2^n + i + 1) : 0 \leq i \leq 2^n - 2\} \). The theorem is proven. \( \Box \)

4. 1-edge hamiltonian

Lemma 4.1. Assume that \( BC(n) = (G_n, w_n, x_n, y_n, z_n) \) for some integer \( n \geq 2 \).

(1) There exists a hamiltonian path \( P_{n}^1 \) of \( G_n \) joining \( w_n \) to \( y_n \).

(2) There exists a hamiltonian path \( P_{n}^2 \) of \( G_n \) joining \( w_n \) to \( z_n \).

(3) There exists a hamiltonian path \( P_{n}^3 \) of \( G_n \) joining \( x_n \) to \( y_n \).

(4) There exists a hamiltonian path \( P_{n}^4 \) of \( G_n \) joining \( x_n \) to \( z_n \).

(5) There exist two disjoint paths \( P_{n}^5 \) and \( P_{n}^6 \) such that

(i) they span \( G_n \),

(ii) \( P_{n}^5 \) joins \( w_n \) and \( x_n \), and

(iii) \( P_{n}^6 \) joins \( z_n \) and \( z_n \).

(6) There exist two disjoint paths \( P_{n}^7 \) and \( P_{n}^8 \) such that

(i) they span \( G_n \),

(ii) \( P_{n}^7 \) joins \( w_n \) and \( z_n \), and

(iii) \( P_{n}^8 \) joins \( x_n \) and \( y_n \).

Proof. We prove this lemma by induction. It is easy to check that the lemma holds in \( BC(2) \). Assume that the lemma holds for \( BC(n) \). By definition, \( BC(n+1) \) is composed of two disjoint copies of \( n \)th brother cells \( BC^1(n) \) and \( BC^2(n) \), a white root \( x_{n+1} \), and a black root \( z_{n+1} \). By induction, \( \{P_{n}^{i+1}\}_{i=1}^{n} \) exists satisfying the lemma for \( BC^1(n) \) and \( \{P_{n}^{i+1}\}_{i=1}^{n} \) exists satisfying the lemma for \( BC^2(n) \).

We then set
\begin{align*}
P_{n+1}^1 &= \{w_{n+1} = w_n, P_{n}^{5,1}, x_{n+1}, z_{n+1}, w_n, (P_{n}^{5,2})^{-1}, w_n, \\
x_n, P_{n}^{6,1}, z_n, x_n, z_{n+1}, z_n, (P_{n}^{6,2})^{-1}, y_n = y_{n+1}\}; \\
P_{n+1}^2 &= \{w_{n+1} = w_n, P_{n}^{5,1}, x_{n+1}, z_n, x_{n+1}, z_n, \\
(P_{n}^{4,2})^{-1}, w_n, (P_{n}^{5,2})^{-1}, z_{n+1}\}; \\
P_{n+1}^3 &= \{x_{n+1}, z_{n+1}, (P_{n}^{5,2})^{-1}, w_n, y_n, (P_{n}^{3,1})^{-1}, \\
x_n, z_{n+1}, w_n, (P_{n}^{5,2})^{-1}, y_n = y_{n+1}\}; \\
P_{n+1}^4 &= \{x_{n+1}, z_{n+1}, (P_{n}^{5,2})^{-1}, w_n, y_n, (P_{n}^{3,1})^{-1}, \\
x_n, z_{n+1}, z_n, (P_{n}^{5,2})^{-1}, z_n, x_n, z_n, z_{n+1}\}; \\
P_{n+1}^5 &= \{w_{n+1} = w_n, P_{n}^{5,1}, y_n, w_n, (P_{n}^{7,2})^{-1}, w_n, z_n, x_{n+1}, z_{n+1}\}; \\
P_{n+1}^6 &= \{y_n, z_n, (P_{n}^{5,2})^{-1}, w_n, (P_{n}^{6,2})^{-1}, x_n, z_{n+1}\}; \\
P_{n+1}^7 &= \{w_{n+1} = w_n, P_{n}^{5,1}, x_{n+1}, z_{n+1}\}; \\
P_{n+1}^8 &= \{x_{n+1}, z_{n+1}, (P_{n}^{6,1})^{-1}, y_n, w_n, (P_{n}^{1,2})^{-1}, y_n = y_{n+1}\};
\end{align*}

The lemma is proven. \( \Box \)
Lemma 4.2. Assume that $BC(n) = (G_n, w_n, x_n, y_n, z_n)$ for some integer $n \geq 2$. Let $e$ be any edge of $BC(n)$. Then at least one of the following properties holds.

1. There exists a hamiltonian path $Q_n^1(e)$ of $G_n - e$ joining $w_n$ to $y_n$.
2. There exists a hamiltonian path $Q_n^2(e)$ of $G_n - e$ joining $w_n$ to $z_n$.
3. There exists a hamiltonian path $Q_n^3(e)$ of $G_n - e$ joining $x_n$ to $y_n$.
4. There exists a hamiltonian path $Q_n^4(e)$ of $G_n - e$ joining $x_n$ to $z_n$.
5. There exist two disjoint paths $Q_n^5(e)$ and $Q_n^6(e)$ such that (i) they span $G_n - e$, (ii) $Q_n^5(e)$ joins $w_n$ and $x_n$, and (iii) $Q_n^6(e)$ joins $y_n$ and $z_n$.
6. There exist two disjoint paths $Q_n^7(e)$ and $Q_n^8(e)$ such that (i) they span $G_n - e$, (ii) $Q_n^7(e)$ joins $w_n$ and $z_n$, and (iii) $Q_n^8(e)$ joins $x_n$ and $y_n$.

Proof. We prove this lemma by induction. It is easy to check that the lemma holds for $BC(2)$. Assume that the lemma holds for $BC(n)$. By definition, $BC(n+1)$ is composed of two disjoint copies of $n$th brother cells $BC^1(n)$ and $BC^2(n)$, a white root $x_{n+1}$, and a black root $z_{n+1}$. Let $e$ be any edge of $BC(n+1)$. Using the symmetrical property of $BC(n+1)$, we may assume that $e$ is $e(1)$, $e(2)$, or $e(3)$, where $e(1)$, $e(2)$, and $e(3)$ are an edge in $BC^1(n)$ or an edge in $BC^2(n)$. By induction, there exists $\{P_{n+1}^{i,j}\}_{j=1}^{8}$ satisfying the lemma for $BC^1(n)$ if $e \notin E(BC^1(n))$ and there exists $\{P_{n+1}^{i,j}\}_{j=1}^{8}$ satisfying the lemma for $BC^2(n)$ if $e \notin E(BC^2(n))$.

Case 1: $e = (x_{n+1}, x_1)$. We set $Q_{n+1}^7(e)$ as $w_{n+1} = w_n^1, P_n^1, y_n^1, w_n^2, P_n^2, x_n^2, z_n^1$ and $Q_{n+1}^8(e)$ as $(x_{n+1}, z_n^1, (P_n^2)^{-1}, x_n^2, y_n^1, z_n^1)$.

Case 2: $e = (z_{n+1}, y_1)$. We set $Q_{n+1}^7(e)$ as $w_{n+1} = w_n^1, P_n^1, y_n^1, w_n^2, P_n^2, x_n^2, z_n^1$ and $Q_{n+1}^8(e)$ as $(y_{n+1} = y_n^1, (P_n^2)^{-1}, x_n^2, z_n^1)$.

Case 3: $e = (y_{n+1}, z_1)$. We set $Q_{n+1}^7(e)$ as $(x_{n+1}, z_n^1, (P_n^2)^{-1}, x_n^2, z_n^1, y_n^1)$.

Case 4: $e$ is in $BC^1(n)$. By induction hypothesis, one of the six properties of the lemma holds for $BC^1(n)$. In the following, we find the corresponding paths that satisfy the lemma for $BC(n+1)$.

1. $Q_{n+1}^7(e) = \{x_{n+1}, w_{n+1}^1, Q_n^1(e), y_n^1, w_n^2, P_n^2, x_n^2, z_n^1\}$ and $Q_{n+1}^8(e) = \{x_{n+1}, z_n^1, (P_n^2)^{-1}, y_n^1, z_n^1\}$.

Theorem 4.1. The brother tree $BT(n)$ is 1-edge hamiltonian for any positive integer $n$ with $n \geq 1$.

Proof. Note that $BT(1)$ is isomorphic to the hypercube $Q_1$. Since $Q_3$ is hamiltonian and edge symmetric, $BT(1)$ is 1-edge hamiltonian. Now we consider $n \geq 2$. By definition, $BT(n)$ is composed of an $(n+1)$th brother cell, denoted by $BC^1(n+1) = (G_{n+1}^1, w_n^1, x_n^1, y_n^1, z_n^1)$. Let $e$ be any edge of $BT(n)$. By the symmetrical property of $BT(n)$, we assume that $e$ is an edge in $G_{n+1}^1$. Using Lemma 4.1, $\{P_n^2\}_{j=1}^{8}$ exists satisfying the lemma for $BC^2(n)$. Using Lemma 4.2, one of the six properties of Lemma 4.2 holds for $BC^1(n+1)$. In the following, we find the corresponding hamiltonian cycle $H_e$ of $BT(n) = e$.

1. $H_e = \{w_n^1, Q_n^1(e), y_n^1, w_n^2, P_n^2, y_n^1, w_{n+1}^1\}$.
2. $H_e = \{w_{n+1}^1, Q_n^1(e), y_n^1, w_n^2, P_n^2, y_n^1, w_{n+1}^1\}$.
3. $H_e = \{x_{n+1}, Q_n^3(e), y_n^1, w_n^2, P_n^2, y_n^1, x_{n+1}\}$.
4. $H_e = \{x_{n+1}, Q_n^3(e), y_n^1, x_{n+1}, y_n^1, x_{n+1}, z_n^1\}$. 

Let $BC(n)$ be a black node of $G_n$. There then exists a hamiltonian path $R_n(c)$ of $G_n - c$ that joins $w_n^1$ to $x_n^1$. Then, we set $R_{n+1}(c)$ as $(w_n^1 = w_n^1, R_n(c), x_n^1, z_n+1, x_{n+1}^1, P_n^3, w_n^2, z_n, x_n^1)$. Suppose that $c = z_n+1$. We set $R_{n+1}(c)$ as $(w_n^1 = w_n^1, P_n^1, y_n^1, y_n^2, P_n^2, z_n, x_n^1)$.

The theorem is proven. An illustration is shown in Fig. 3. □

5. $1_p$-hamiltonian

**Lemma 5.1.** Assume that $n$ is an integer with $n \geq 2$. Let $BC(n) = (G_n, w_n, x_n, y_n, z_n)$. Suppose that $c$ is any node of $G_n$. There then exists a hamiltonian path $R_n(c)$ of $G_n - c$ such that $R_n(c)$ joins $y_n$ to $z_n$ if $c$ is a white node, and $R_n(c)$ joins $w_n$ to $x_n$ if $c$ is a black node.

**Proof.** We prove this lemma by induction. It is easy to check that the lemma holds for $BC(2)$. Assume that the lemma holds for $BC(n)$. By definition, $BC(n+1)$ is composed of two disjoint copies of $n$th brother cells $BC^1(n) = (G_n, w_n, x_n, y_n, z_n)$ and $BC^2(n) = (G_n, w_n, x_n, y_n, z_n)$, a white root $x_n$, and a black root $z_{n+1}$. We only prove the case that $c$ is a black node. Using the symmetrical property of $BC(n+1)$, we may assume that $c$ is a node in $BC^1(n)$ or $c = z_{n+1}$.

Using Lemma 4.1, there exists $\{P_n^1\}_{i=1}^8$ for $BC^1(n)$ if $c \notin V(BC^1(n))$ and $\{P_n^2\}_{i=1}^8$ for $BC^2(n)$ if $c \notin V(BC^2(n))$.

Suppose that $c$ is in $BC^1(n)$. By induction, there exists a hamiltonian path $R_n^1(c)$ of $G_n - c$ that joins $w_n$ to $x_n$. Then, we set $R_{n+1}(c)$ as $(w_n = w_n, R_n^1(c), x_n, z_{n+1}, x_{n+1}, P_n^3, z_n, w_n)$. Suppose that $c = z_{n+1}$. We set $R_{n+1}(c)$ as $(w_n = w_n, P_n, y_n, P_n^2, z_n, x_n)$. The lemma is proven. □

**Lemma 5.2.** Assume that $n$ is an integer with $n \geq 2$. Let $BC(n) = (G_n, w_n, x_n, y_n, z_n)$. Let $c$ be a white node of $G_n$ and $d$ be a black node of $G_n$. Then at least one of the following properties holds.

1. There exists a hamiltonian path $S_n^1(c, d)$ of $G_n - \{c, d\}$ joining $w_n$ to $y_n$.
2. There exists a hamiltonian path $S_n^2(c, d)$ of $G_n - \{c, d\}$ joining $w_n$ to $z_n$.
3. There exists a hamiltonian path $S_n^3(c, d)$ of $G_n - \{c, d\}$ joining $x_n$ to $y_n$.
4. There exists a hamiltonian path $S_n^4(c, d)$ of $G_n - \{c, d\}$ joining $x_n$ to $z_n$.
5. There exist two disjoint paths $S_n^5(c, d)$ and $S_n^6(c, d)$ such that (i) they span $G_n - \{c, d\}$, (ii) $S_n^5(c, d)$ joins $w_n$ and $x_n$, and (iii) $S_n^6(c, d)$ joins $y_n$ and $z_n$. 
(6) There exist two disjoint paths \( S_{n}(c, d) \) and \( S'_{n}(c, d) \) such that (i) they span \( G_n = \{c, d\} \), (ii) \( S'_{n}(c, d) \) joins \( w_{n} \) and \( z_{n} \), and (iii) \( S_{n}(c, d) \) joins \( x_{n} \) and \( y_{n} \).

**Proof.** We prove this lemma by induction. It is easy to check that the lemma holds for \( BC(2) \). Assume that the lemma holds for \( BC(n) \). By definition, \( BC(n + 1) \) is composed of two disjoint copies of \( n \)-th brother cells \( BC^1(n) \) and \( BC^2(n) \), a white root \( x_{n+1} \), and a black root \( z_{n+1} \). Using the symmetrical property of \( BC(n + 1) \), we may assume that one of the following cases holds:

1. \( c = x_{n+1} \) and \( d = z_{n+1} \),
2. \( c = x_{n+1} \) and \( d \in V(BC^1(n)) \),
3. \( c \in V(BC(n)) \) and \( d = z_{n+1} \),
4. \( c \in V(BC(n)) \) and \( d \in V(BC^2(n)) \) or \( c, d \in V(BC^1(n)) \).

In the following, we find the corresponding path(s) for each case.

**Case 1:** We set \( S_{n+1}^1(c, d) = \langle w_{n+1} = w_n, x_1, x_n, y_2, y_n \rangle \).

**Case 2:** With Lemma 5.1, we set \( S_{n+1}^1(c, d) = \langle w_{n+1} = w_n, p_{n}^{1,2}, y_1 \rangle \).

**Case 3:** With Lemma 5.1, we set \( S_{n+1}^2(c, d) = \langle x_{n+1}, z_{n+1}, R_{n}^{1}(d), x_1, x_n, z_{n+1}, y_2, y_n \rangle \).

**Case 4:** With Lemma 5.1, we set \( S_{n+1}^2(c, d) = \langle x_{n+1}, z_{n+1}, R_{n}^{1}(c), x_1, x_n, y_2, y_n \rangle \).

**Case 5:** By induction hypothesis, one of the six properties of the lemma holds for \( BC^1(n) \). Note that the endpoints of \( S_n^1(c, d) \) are the same as the endpoints of \( Q_n^1(c) \) stated in Lemma 4.2. With a similar argument for the case (4) in Lemma 4.2, we can prove that the lemma is true for this case.

Hence, the lemma is proven.

**Theorem 5.1.** The brother tree \( BT(n) \) is \( 1_p \)-hamiltonian for any positive integer \( n \) with \( n \geq 1 \).

**Proof.** This theorem can be obtained with a similar argument of Theorem 4.1.

**6. Conclusion**

In this paper, we propose a family of bipartite graphs called brother trees, denoted by \( BT(n) \). \( BT(n) \) is a planar, bipartite, 3-regular graph, and the number of nodes in \( BT(n) \) is \( 6 \cdot 2^n - 4 \). Moreover, we prove that \( BT(n) \) is optimal 1-edge hamiltonian and 1-\( p \)-hamiltonian, and the diameter of \( BT(n) \) is \( 2n + 1 \).

Let \( G \) be a graph with \( p \) nodes and with maximal degree \( d \). The famous Moore bound [2] states that the diameter \( D(G) \) is at least \( \log_{d-1}(p - 2/d) \). Thus the diameter of \( BT(n) \) is about 2 times the Moore bound. It is interesting to find other optimal 1-edge hamiltonian and \( 1_p \)-hamiltonian bipartite graphs with smaller diameters.

We also note that the complete binary tree is one of the most important architectures for interconnection networks [4]. A lot of complete binary tree variations have been proposed. Because the brother tree is composed of several complete binary trees, we believe that the brother tree is another candidate for interconnection networks.

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**References**