

Anisotropic power-law inflation for a two scalar fields model

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A special class of Bianchi type I expanding solutions in a string motivated theory with a single scalar field has been speculated to break the cosmic no-hair theorem that will not approach the late time isotropic expanding solution. We will show by a new perturbation approach that an unstable mode for the inflationary solutions exists when an additional phantom field is introduced. The result indicates that the existence of an unstable mode is closely related to the extra fields that could be present during the very early Universe.

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I. INTRODUCTION

The inflationary scenario offers a natural explanation for a number of important phenomenons of cosmic microwave background radiation (CMB) that has been very well confirmed by the Wilkinson Microwave Anisotropy Probe [1,2]. One of the remarkable predictions of inflation is the cosmic no-hair conjecture which states that any classical hair should disappear once the vacuum energy dominates.

The field equations of any gravitational system with a cosmological constant Λ can always be written as

$$G_{\mu\nu} = T_{\mu\nu} - \Lambda g_{\mu\nu}. \quad (1.1)$$

The Einstein tensor $G_{\mu\nu}$ on the left-hand-side of the above equation represents the geometric impact of the gravitational effect driven by the energy momentum tensor $T_{\mu\nu}$ on the right-hand-side of the above equation.

Gibbons and Hawking [3], and Hawking and Moss [4] conjectured that all models with a positive cosmological constant will approach a late time de Sitter space. This is later became known as the cosmic no-hair theorem for the Einstein gravity. Partial proof was given by Robert Wald [5] which shows clearly that any model with a positive cosmological constant will drive the late-time evolution toward de Sitter spacetime, at least locally, for all non-type IX Bianchi spaces provided that the matter sources obey both the dominant energy condition (DEC), $T_{\mu\nu}t^\mu t^\nu \geq 0$, and the strong-energy condition (SEC), $(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)t^\mu t^\nu \geq 0$, for all timelike vectors t^μ [5]. Here $T_{\mu\nu}$ and T denote the energy momentum tensor and its trace for all the fields coupled to the gravitational system. The type IX Bianchi space behaves similarly if Λ is sufficiently large [5].

A number of studies of the cosmic no-hair theorems have been shown to support the existence of certain constraints on the field parameters in order for the cosmic hair to be absent [5–14]. It is also known, however, that counterexamples exist where these energy conditions do not hold exactly [15–17]. Many of these solutions have later been shown to be unstable [10,18–20] which appear to

support Hawking's no-hair conjecture. In any case, it is important to check carefully to find out whether all existing claims that anisotropically expanding solutions are stable are true or not. These investigations may further our vision and understanding on the limit and constraints relative to the evolutionary Universe.

Many of the studies focus on the effect of the higher derivative gravity theories. Relative higher-order theories can also be found in Refs. [21–52]. For example, we have been able to show that the inflationary solutions found [40] in the Bianchi type II and type VI spaces are in fact unstable in the presence of anisotropic perturbations [43,46]. Note that the Bianchi type II solutions (and some Bianchi type I inflating solutions) were also found to be unstable by Barrow and Hervik in Ref. [41].

Recently, another example of anisotropic inflationary solutions has been found that seems to provide one more counterexample to the cosmic no-hair conjecture [53–58]. Indeed, it was shown that in the presence of a vector field coupled with the inflaton, there could be a small anisotropy in the expansion rate, which never decays during inflation. Furthermore, the anisotropic inflation is also shown to be an attractor solution [59]. In addition, analytic solutions are also found in an anisotropic inflationary model with a single scalar field coupled to the system motivated by supergravity theory [59]. Anisotropic hair seems to persist in this model even when a large cosmological constant is not present. In the hope that the no-hair conjecture could survive one way or the other, we wish to find out whether unstable modes could exist in other perturbation directions not considered in this model. For example, the presence of additional fields could possibly change the stability properties of the expanding solutions.

In fact, we will show that a set of expanding solutions can also be found in a model with a two scalar field model [60–62] similar to the one scalar model [59]. For the solutions to be expanding solutions, three independent inequalities will be shown to put strong constraints on the system. In particular, we will show that the inflationary solutions requiring stronger constraints on the field parameters will drive the expanding solutions to collapse as promised.

Note that the solutions we found will be shown to be the solutions found in Ref. [59] if we set $\rho_2 = 0$ and $\lambda_2 \rightarrow \infty$ together. The result will be that the unstable mode may not exist in the one scalar model, but the presence of additional coupling apparently changes the stability pattern of the expanding solutions. This result will offer a new approach to the stability analysis in similar systems.

This paper will be organized as follows: (i) a brief review of the motivation of this research is given in Sec. I, (ii) in Sec. II, a two scalar model will be introduced and analyzed, (iii) anisotropic Bianchi type I solutions will be solved in Sec. III, (iv) we will show that the presence of the additional scalar field does affect the stability of the anisotropic expanding solutions. In particular, we will show that the system is unstable in the inflationary phase, (v) finally, concluding remarks are given in Sec. V.

II. TWO SCALAR MODEL

We will extend the one scalar model studied by Kanno *et al.*'s model [59] to a cosmological model with two scalar fields that is known as the *quintom model* [60–62] given by the action:

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{C_1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{C_2}{2} (\partial_\mu \psi)(\partial^\mu \psi) - V(\phi, \psi) - \frac{1}{4} f^2(\phi, \psi) F_{\mu\nu} F^{\mu\nu} \right]. \quad (2.1)$$

Here $V(\phi, \psi)$ is a twice continuously differentiable function, M_p is the Planck mass. The fields ϕ and ψ are the scalar and the phantom fields, respectively, if the signs of the constants are chosen as $C_1 > 0$ and $C_2 < 0$ [61]. We will set the constants as $C_1 = 1$, $C_2 = -1$, and choose $V(\phi, \psi) = V_1(\phi) + V_2(\psi)$ [62] along with $f^2(\phi, \psi) = f_1^2(\phi) f_2^2(\psi)$. As a result, the field equations of this model can be shown to be

$$\frac{\partial}{\partial x^\mu} \left(\sqrt{-g} f^2(\phi, \psi) F^{\mu\nu} \right) = 0, \quad (2.2)$$

$$\ddot{\phi} = -3H\dot{\phi} - \partial_\phi V(\phi, \psi) - \frac{1}{2} f(\phi, \psi) \partial_\phi f(\phi, \psi) F_{\mu\nu} F^{\mu\nu}, \quad (2.3)$$

$$\ddot{\psi} = -3H\dot{\psi} + \partial_\psi V(\phi, \psi) + \frac{1}{2} f(\phi, \psi) \partial_\psi f(\phi, \psi) F_{\mu\nu} F^{\mu\nu}, \quad (2.4)$$

$$M_p^2 (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) - \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} g_{\mu\nu} \partial^\sigma \phi \partial_\sigma \phi + \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial^\sigma \psi \partial_\sigma \psi + g_{\mu\nu} [V(\phi, \psi) + \frac{1}{4} f^2(\phi, \psi) F^{\rho\sigma} F_{\rho\sigma}] - f^2(\phi, \psi) F_{\mu\gamma} F_\nu^\gamma = 0. \quad (2.5)$$

For the Bianchi type I metric given by

$$ds^2 = -dt^2 + \exp[2\alpha(t) - 4\sigma(t)] dx^2 + \exp[2\alpha(t) + 2\sigma(t)] (dy^2 + dz^2), \quad (2.6)$$

with gauge field and scalar fields take the form $A_\mu = (0, A_x(t), 0, 0)$, $\phi = \phi(t)$, and $\psi = \psi(t)$ which are consistently coupled to each other on the Bianchi type I metric space. The solution to Eq. (2.2) can be integrated immediately to give

$$\dot{A}_x(t) = f^{-2}(\phi, \psi) \exp[-\alpha - 4\sigma] p_A, \quad (2.7)$$

with p_A a constant of integration [59]. Therefore, Eqs. (2.3) and (2.4) become

$$\ddot{\phi} = -3\dot{\alpha}\dot{\phi} - \frac{\partial V(\phi, \psi)}{\partial \phi} + f^{-3}(\phi, \psi) \frac{\partial f(\phi, \psi)}{\partial \phi} \times \exp[-4\alpha - 4\sigma] p_A^2, \quad (2.8)$$

$$\ddot{\psi} = -3\dot{\alpha}\dot{\psi} + \frac{\partial V(\phi, \psi)}{\partial \psi} - f^{-3}(\phi, \psi) \frac{\partial f(\phi, \psi)}{\partial \psi} \times \exp[-4\alpha - 4\sigma] p_A^2, \quad (2.9)$$

respectively. As a result, we can obtain the following set of field equations:

$$\dot{\alpha}^2 = \dot{\sigma}^2 + \frac{1}{3M_p^2} \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \dot{\psi}^2 + V(\phi, \psi) + \frac{1}{2} f^{-2}(\phi, \psi) \exp[-4\alpha - 4\sigma] p_A^2 \right], \quad (2.10)$$

$$\ddot{\alpha} = -3\dot{\alpha}^2 + \frac{1}{M_p^2} V(\phi, \psi) + \frac{1}{6M_p^2} f^{-2}(\phi, \psi) \times \exp[-4\alpha - 4\sigma] p_A^2, \quad (2.11)$$

$$\ddot{\sigma} = -3\dot{\alpha}\dot{\sigma} + \frac{1}{3M_p^2} f^{-2}(\phi, \psi) \exp[-4\alpha - 4\sigma] p_A^2. \quad (2.12)$$

III. ANISOTROPIC POWER-LAW SOLUTIONS

For the purpose of our stability analysis, we will choose the exponential potentials of the form

$$V(\phi, \psi) = V_{01} \exp\left[\frac{\lambda_1}{M_p} \phi\right] + V_{02} \exp\left[\frac{\lambda_2}{M_p} \psi\right], \quad (3.1)$$

with the gauge kinetic function of the form

$$f(\phi, \psi) = f_0 \exp\left[\frac{\rho_1}{M_p} \phi\right] \exp\left[\frac{\rho_2}{M_p} \psi\right]. \quad (3.2)$$

Here V_{0i} , λ_i , and ρ_i are constant field parameters. We will try to find expanding or inflationary power-law solutions of the following form [59]:

$$\begin{aligned} \alpha &= \zeta \log(t); \\ \sigma &= \eta \log(t); \\ \frac{\phi}{M_p} &= \xi_1 \log(t) + \phi_0; \\ \frac{\psi}{M_p} &= \xi_2 \log(t) + \psi_0. \end{aligned} \quad (3.3)$$

For convenience, we will also define the following new variables:

$$u = \frac{V_{01} \exp[\lambda_1 \phi_0]}{M_p^2}, \quad (3.4)$$

$$v = \frac{V_{02} \exp[\lambda_2 \psi_0]}{M_p^2}, \quad (3.5)$$

$$l = \frac{p_A^2 f_0^{-2} \exp[-2(\rho_1 \phi_0 + \rho_2 \psi_0)]}{M_p^2} \quad (3.6)$$

as a set of new positive parameters. With the ansatz given by Eq. (3.3), the field Eqs. (2.8), (2.9), (2.10), (2.11), and (2.12) become a set of algebraic equations:

$$-\xi_1 + 3\zeta\xi_1 + \lambda_1 u - \rho_1 l = 0, \quad (3.7)$$

$$-\xi_2 + 3\zeta\xi_2 - \lambda_2 v + \rho_2 l = 0, \quad (3.8)$$

$$-\zeta^2 + \eta^2 + \frac{(\xi_1^2 - \xi_2^2)}{6} + \frac{(u+v)}{3} + \frac{l}{6} = 0, \quad (3.9)$$

$$-\zeta + 3\zeta^2 - (u+v) - \frac{l}{6} = 0, \quad (3.10)$$

$$-\eta + 3\zeta\eta - \frac{l}{3} = 0. \quad (3.11)$$

Note that we have assumed the following constraint equations:

$$\lambda_1 \xi_1 = -2, \quad (3.12)$$

$$\lambda_2 \xi_2 = -2, \quad (3.13)$$

$$\rho_1 \xi_1 + \rho_2 \xi_2 + 2\zeta + 2\eta = 1, \quad (3.14)$$

such that the potential term $V(\phi, \psi)$ and the gauge coupling terms in the Hamiltonian constraint (2.12) are all proportional to $1/t^2$. Setting $\xi_1 = -2/\lambda_1$ and $\xi_2 = -2/\lambda_2$, Eq. (3.14) can be written as

$$\eta + \zeta = \frac{1}{2} + \frac{\rho_1}{\lambda_1} + \frac{\rho_2}{\lambda_2}, \quad (3.15)$$

In addition, we can also show that this set of equations leads to a polynomial equation of ζ as

$$\begin{aligned} &(3\zeta - 1)[\{6\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)\}\zeta \\ &\quad - [4(\lambda_1\rho_2 + \lambda_2\rho_1)(2\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) \\ &\quad + \lambda_1^2\lambda_2^2 + 8(\lambda_2^2 - \lambda_1^2)]] = 0. \end{aligned} \quad (3.16)$$

There is hence a trivial solution $\zeta = 1/3$ that leads to the trivial constraints $u = v = l = 0$. In addition, there is also a nontrivial solution given by

$$\zeta = \frac{4(\lambda_1\rho_2 + \lambda_2\rho_1)(2\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) + \lambda_1^2\lambda_2^2 + 8(\lambda_2^2 - \lambda_1^2)}{6\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}. \quad (3.17)$$

As a result, the parameters η , u , v , l can be parametrized as functions of λ_i and ρ_i given by

$$\eta = \frac{\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\lambda_2^2 - \lambda_1^2)}{3\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}, \quad (3.18)$$

$$u = \frac{\Omega \times [\lambda_2^2(\lambda_1\rho_1 + 2\rho_1^2 + 2) + 2\lambda_1\lambda_2\rho_1\rho_2 + 4(\lambda_1\rho_1 + \lambda_2\rho_2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \quad (3.19)$$

$$v = \frac{\Omega \times [\lambda_1^2(\lambda_2\rho_2 + 2\rho_2^2 - 2) + 2\lambda_1\lambda_2\rho_1\rho_2 - 4(\lambda_1\rho_1 + \lambda_2\rho_2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \quad (3.20)$$

$$l = \frac{\Omega \times [\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\lambda_2^2 - \lambda_1^2)]}{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)]^2}, \quad (3.21)$$

with $\Omega \equiv 4(\lambda_1\rho_2 + \lambda_2\rho_1)(\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) - \lambda_1^2\lambda_2^2 + 8(\lambda_2^2 - \lambda_1^2)$. Note that the solutions found above become the same set of solutions found for the one scalar model if we take the limit $\rho_2 \rightarrow 0$ and $\lambda_2 \rightarrow \infty$ [59].

We can also define the average slow-roll parameter given by [59]

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{6\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}{4(\lambda_1\rho_2 + \lambda_2\rho_1)(2\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) + \lambda_1^2\lambda_2^2 + 8(\lambda_2^2 - \lambda_1^2)}. \quad (3.22)$$

In addition, the anisotropy is given by

$$\frac{\Sigma}{H} \equiv \frac{\dot{\sigma}}{\dot{\alpha}} = \frac{2[\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1) - 4(\lambda_2^2 - \lambda_1^2)]}{4(\lambda_1\rho_2 + \lambda_2\rho_1)(2\lambda_1\lambda_2 + 3\lambda_1\rho_2 + 3\lambda_2\rho_1) + \lambda_1^2\lambda_2^2 + 8(\lambda_2^2 - \lambda_1^2)}. \quad (3.23)$$

Consequently, there is a relation between the average slow-roll parameter and the anisotropy given by comparing Eq. (3.22) with Eq. (3.23):

$$\frac{\Sigma}{H} = \frac{1}{3}I\varepsilon, \quad (3.24)$$

where

$$I = 3\eta = 1 - \frac{4(\lambda_2^2 - \lambda_1^2)}{\lambda_1\lambda_2(\lambda_1\lambda_2 + 2\lambda_1\rho_2 + 2\lambda_2\rho_1)}. \quad (3.25)$$

Hence, $\lambda_2 > \lambda_1$ is required if the constraint $0 < I < 1$ is to be observed [59].

IV. STABILITY ANALYSIS OF THE EXPANDING SOLUTIONS

Perturbing the field Eqs. (2.8), (2.9), (2.10), (2.11), and (2.12), we can obtain the following set of perturbation equations:

$$\begin{aligned} \delta\ddot{\phi} = & -3\frac{\zeta}{t}\delta\dot{\phi} - 3\frac{\xi_1 M_p}{t}\delta\dot{\alpha} - \frac{\lambda_1^2 u}{t^2}\delta\phi \\ & - \frac{\rho_1 M_p l}{t^2} \left[\frac{2}{M_p}(\rho_1\delta\phi + \rho_2\delta\psi) + 4(\delta\alpha + \delta\sigma) \right], \end{aligned} \quad (4.1)$$

$$\begin{aligned} \delta\ddot{\psi} = & -3\frac{\zeta}{t}\delta\dot{\psi} - 3\frac{\xi_2 M_p}{t}\delta\dot{\alpha} + \frac{\lambda_2^2 v}{t^2}\delta\psi \\ & + \frac{\rho_2 M_p l}{t^2} \left[\frac{2}{M_p}(\rho_1\delta\phi + \rho_2\delta\psi) + 4(\delta\alpha + \delta\sigma) \right], \end{aligned} \quad (4.2)$$

$$\begin{aligned} \frac{2\zeta}{t}\delta\dot{\alpha} = & 2\frac{\eta}{t}\delta\dot{\sigma} + \frac{\xi_1}{3M_p t}\delta\dot{\phi} - \frac{\xi_2}{3M_p t}\delta\dot{\psi} + \frac{\lambda_1 u}{3M_p t^2}\delta\phi \\ & + \frac{\lambda_2 v}{3M_p t^2}\delta\psi - \frac{l}{6t^2} \left[\frac{2}{M_p}(\rho_1\delta\phi + \rho_2\delta\psi) \right. \\ & \left. + 4(\delta\alpha + \delta\sigma) \right], \end{aligned} \quad (4.3)$$

$$\begin{aligned} \delta\ddot{\alpha} = & -6\frac{\zeta}{t}\delta\dot{\alpha} + \frac{\lambda_1 u}{M_p t^2}\delta\phi + \frac{\lambda_2 v}{M_p t^2}\delta\psi \\ & - \frac{l}{6t^2} \left[\frac{2}{M_p}(\rho_1\delta\phi + \rho_2\delta\psi) + 4(\delta\alpha + \delta\sigma) \right], \end{aligned} \quad (4.4)$$

$$\begin{aligned} \delta\ddot{\sigma} = & -3\frac{\zeta}{t}\delta\dot{\sigma} - 3\frac{\eta}{t}\delta\dot{\alpha} - \frac{l}{3t^2} \left[\frac{2}{M_p}(\rho_1\delta\phi + \rho_2\delta\psi) \right. \\ & \left. + 4(\delta\alpha + \delta\sigma) \right]. \end{aligned} \quad (4.5)$$

Instead of taking the exponential perturbation of fields as $\delta\alpha = \alpha_0 \exp[nt]$ that is not compatible with the power-law solutions here, we will take the power-law perturbation of fields defined by $\delta\alpha = At^n$, $\delta\sigma = Bt^n$, $\delta\phi = M_p Ct^n$, $\delta\psi = M_p Dt^n$ [59]. As a result, the above set of perturbation equations becomes a set of algebraic equations:

$$\begin{aligned} \left(\frac{6n}{\lambda_1} - 4\rho_1 l \right) A - 4\rho_1 l B - [n(n-1) + 3\zeta n \\ + \lambda_1^2 u + 2\rho_1^2 l] C - 2\rho_1 \rho_2 l D = 0, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \left(\frac{6n}{\lambda_2} + 4\rho_2 l \right) A + 4\rho_2 l B - 2\rho_1 \rho_2 l C - [n(n-1) \\ + 3\zeta n - \lambda_2^2 v - 2\rho_2^2 l] D = 0, \end{aligned} \quad (4.7)$$

$$\begin{aligned} - \left(\frac{2l}{3} + 2\zeta n \right) A - \left(\frac{2l}{3} + 2\eta n \right) B \\ + \left(-\frac{2n}{3\lambda_1} + \frac{u\lambda_1}{3} - \frac{l\rho_1}{3} \right) C \\ - \left(-\frac{2n}{3\lambda_2} - \frac{v\lambda_2}{3} + \frac{l\rho_2}{3} \right) D = 0, \end{aligned} \quad (4.8)$$

$$\begin{aligned} - \left[n(n-1) + 6\zeta n + \frac{2l}{3} \right] A - \frac{2l}{3} B + \left(u\lambda_1 - \frac{l\rho_1}{3} \right) C \\ + \left(v\lambda_2 - \frac{l\rho_2}{3} \right) D = 0, \end{aligned} \quad (4.9)$$

$$\begin{aligned}
& - \left(3\eta n + \frac{4l}{3} \right) A - \left[n(n-1) + 3\zeta n + \frac{4l}{3} \right] B \\
& - \frac{2l\rho_1}{3} C - \frac{2l\rho_2}{3} D = 0,
\end{aligned} \tag{4.10}$$

with the identities $\lambda_1 \xi_1 = \lambda_2 \xi_2 = -2$ used to write the equations above as functions of λ_i and ρ_i . These equations can be written as a matrix equation:

$$\mathcal{D} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0, \tag{4.11}$$

with

$$\begin{aligned}
A_{11} &= \left(\frac{6n}{\lambda_1} - 4\rho_1 l \right); \\
A_{12} &= -4\rho_1 l; \\
A_{13} &= -[n(n-1) + 3\zeta n + \lambda_1^2 u + 2\rho_1^2 l]; \\
A_{14} &= -2\rho_1 \rho_2 l;
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
A_{21} &= \left(\frac{6n}{\lambda_2} + 4\rho_2 l \right); \\
A_{22} &= 4\rho_2 l; \\
A_{23} &= -2\rho_1 \rho_2 l; \\
A_{24} &= -[n(n-1) + 3\zeta n - \lambda_2^2 v - 2\rho_2^2 l];
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
A_{31} &= -\left[n(n-1) + 6\zeta n + \frac{2l}{3} \right]; \\
A_{32} &= -\frac{2l}{3}; \\
A_{33} &= \left(u\lambda_1 - \frac{l\rho_1}{3} \right); \\
A_{34} &= \left(v\lambda_2 - \frac{l\rho_2}{3} \right);
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
A_{41} &= -\left(3\eta n + \frac{4l}{3} \right); \\
A_{42} &= -\left[n(n-1) + 3\zeta n + \frac{4l}{3} \right]; \\
A_{43} &= -\frac{2l\rho_1}{3}; \\
A_{44} &= -\frac{2l\rho_2}{3}.
\end{aligned} \tag{4.15}$$

It is known that nontrivial solutions of the Eq. (4.11) exist only when

$$\det \mathcal{D} = 0. \tag{4.16}$$

Note that we can write the Eq. (4.16) as a polynomial equation of n :

$$\begin{aligned}
& b_8 n^8 + b_7 n^7 + b_6 n^6 + b_5 n^5 + b_4 n^4 + b_3 n^3 \\
& + b_2 n^2 + b_1 n = 0,
\end{aligned} \tag{4.17}$$

along with a vanishing constant term $b_0 = 0$ that implies the existence of a trivial solution $n = 0$. In addition, we can also show that $b_8 = 1$. Therefore, we need to solve the following polynomial equation with nontrivial coefficients b_i ,

$$\begin{aligned}
f(n) &= n^7 + b_7 n^6 + b_6 n^5 + b_5 n^4 + b_4 n^3 + b_3 n^2 \\
& + b_2 n + b_1 = 0,
\end{aligned} \tag{4.18}$$

for the perturbation solutions. Note that the coefficient b_1 can also be calculated and written as

$$\begin{aligned}
b_1 &= -2\nu l [\lambda_1^2 \lambda_2^2 (5\zeta - \eta - 1)u + 2\lambda_1 \lambda_2 (\lambda_1 \rho_2 + \lambda_2 \rho_1) \\
& \times (3\zeta - 3\eta - 1)u + 8\lambda_2 \rho_1^2 \rho_2 (3\zeta - 3\eta - 1) \\
& + 4(\lambda_1^2 - \lambda_2^2)u - 16\lambda_1^{-1} \lambda_2 \rho_1 \rho_2],
\end{aligned} \tag{4.19}$$

For the solutions found in this paper to be expanding solutions, $\zeta + \eta$ and $\zeta - 2\eta$ must be all positive. In addition, the parameters u , ν , and l are all defined as positive parameters. These will put forward a set of constraints on the field parameters.

It is straightforward to show that, if the parameters λ_i , ρ_i are all positive, $\zeta + \eta = 1/2 + (\rho_1 \lambda_2 + \rho_2 \lambda_1)/(\lambda_1 \lambda_2) > 0$ following Eq. (3.15). In addition, it can also be shown that $\zeta - 2\eta > 0$ if

$$4(\rho_1 \lambda_2 + \rho_2 \lambda_1)^2 + 8\lambda_2^2 > \lambda_1^2 \lambda_2^2 + 8\lambda_1^2. \tag{4.20}$$

It is also straightforward to show that $u > 0$ if $\Omega > 0$ which implies that

$$\begin{aligned}
& 4(\rho_1 \lambda_2 + \rho_2 \lambda_1)(\lambda_1 \lambda_2 + 3\rho_1 \lambda_2 + 3\rho_2 \lambda_1) \\
& + 8\lambda_2^2 > \lambda_1^2 \lambda_2^2 + 8\lambda_1^2.
\end{aligned} \tag{4.21}$$

Note that the inequality (4.21) holds if the inequality (4.20) holds. Therefore, the above inequality is a redundant inequality. Similarly, $l > 0$ and $\nu > 0$ imply, respectively, that

$$\lambda_1 \lambda_2 (\lambda_1 \lambda_2 + 2\rho_1 \lambda_2 + 2\rho_2 \lambda_1) + 4\lambda_1^2 > 4\lambda_2^2, \tag{4.22}$$

$$\lambda_1^2 \rho_2 (\lambda_2 + 2\rho_2) + 2\lambda_1 \lambda_2 \rho_1 \rho_2 > 2\lambda_1^2 + 4(\lambda_1 \rho_1 + \lambda_2 \rho_2). \tag{4.23}$$

Hence, we have three totally independent inequalities to be observed: (4.20), (4.22), and (4.23). Note further that we can write two of the inequalities, (4.20) and (4.23), in a more comprehensive form as

$$2E + 1 > 2E - 8F - 1 > 0 \quad (4.24)$$

with

$$E = \frac{\rho_1 \lambda_2 + \rho_2 \lambda_1}{\lambda_1 \lambda_2} \quad \text{and} \quad (4.25)$$

$$F = \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1 \lambda_2 (\lambda_1 \lambda_2 + 2\rho_1 \lambda_2 + 2\rho_2 \lambda_1)} > -\frac{1}{4}.$$

Furthermore, if the expanding solutions can be used to represent inflationary solutions, it will require that $\zeta - 2\eta \gg 1$ and $\zeta + \eta \gg 1$. As a result, $\zeta + \eta \gg 1$ implies that

$$\rho_1 \lambda_2 + \rho_2 \lambda_1 \gg \lambda_1 \lambda_2. \quad (4.26)$$

In addition, $\zeta - 2\eta \gg 1$ together with the constraints derived above from $\zeta + \eta \gg 1$ lead to the following constraint:

$$(\rho_1 \lambda_2 + \rho_2 \lambda_1)^2 \gg \lambda_1^2 - \lambda_2^2. \quad (4.27)$$

If we write $b_1 = -2\nu l(k_1 u + k_2)$, with the k_1 term representing the part linear in u and the k_2 term representing the part independent of u , k_i can be written as

$$k_1 = \left(5E - 8F - \frac{3}{2}\right) \lambda_1^2 \lambda_2^2$$

$$+ 2\left(3E - 8F - \frac{5}{2}\right) \lambda_1 \lambda_2 (\rho_1 \lambda_2 + \rho_2 \lambda_1) + 4(\lambda_1^2 - \lambda_2^2)$$

$$+ \lambda_1 \lambda_2 (\lambda_1 \lambda_2 + 2\rho_1 \lambda_2 + 2\rho_2 \lambda_1) > \left(5E - 8F - \frac{3}{2}\right) \lambda_1^2 \lambda_2^2$$

$$+ 2\left(3E - 8F - \frac{5}{2}\right) \lambda_1 \lambda_2 (\rho_1 \lambda_2 + \rho_2 \lambda_1), \quad (4.28)$$

with the last inequality following from the inequality (4.27). With the help of the inequality (4.24), we can further write the bound of k_1 as

$$k_1 > \left(3E - \frac{1}{2}\right) \lambda_1^2 \lambda_2^2 + 2\left(E - \frac{3}{2}\right) \lambda_1 \lambda_2 (\rho_1 \lambda_2 + \rho_2 \lambda_1). \quad (4.29)$$

In addition, k_2 takes the form

$$k_2 = \left(3E - 8F - \frac{3}{2}\right) \lambda_1 \rho_1 - 2 > \left(E - \frac{1}{2}\right) \lambda_1 \rho_1 - 2. \quad (4.30)$$

Here we have also used the inequality (4.24) to derive the last inequality.

The above constraint equations, derived from the positivity of various field parameters can be easily met if $\rho_1 \lambda_2 + \rho_2 \lambda_1 \gg \lambda_1 \lambda_2 \sim O(1)$. In such cases, the coefficients k_i are all positive and hence b_1 is always negative for inflationary solutions.

Note that the polynomial equation $f(n)$ is a degree 7 polynomial equation that goes to infinity when $n \rightarrow \infty$. Since we have shown that $b_1 < 0$ for inflationary solutions, there is at least a positive root to this polynomial equation.

Indeed, if $f(0) = b_1 < 0$ and $f(\infty) \rightarrow \infty$, there is at least a point of intersection where the curve $f(n)$ crosses the positive n axis on the n - $f(n)$ plane. Hence, we reach the conclusion that a positive mode with $n > 0$ exists for inflationary solutions. The positive mode represents an unstable mode of the perturbation equations. Therefore, we have shown that this set of inflationary solutions is unstable.

V. CONCLUSION

With the limit $\lambda_2 \rightarrow \infty$, $\rho_2 \rightarrow 0$, we can write the perturbation equations for the one scalar field model as functions of λ_1 and ρ_1 :

$$\tilde{\mathcal{D}} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \end{bmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0. \quad (5.1)$$

A nontrivial solution of the Eq. (5.1) exists only when

$$\det \tilde{\mathcal{D}} = 0. \quad (5.2)$$

This equation can be shown to give a degree 6 polynomial equation of n :

$$\tilde{f}(n) = -n^6 + c_5 n^5 + c_4 n^4 + c_3 n^3 + c_2 n^2 + c_1 n = 0, \quad (5.3)$$

with $c_6 = -1$ and $c_0 = 0$ explicitly shown in the above equation. In addition, the coefficient c_1 can also be calculated and written as

$$c_1 = -2\nu l[(5\zeta - \eta - 1)\lambda_1^2 + 2(3\zeta - 3\eta - 1)\lambda_1 \rho_1 - 4]. \quad (5.4)$$

The leading coefficient of the n^6 term changes sign for the one scalar model solutions. Therefore, the unstable model no longer persists here. It is apparent that the sign change of the leading order term coefficient plays an important role once the phantom field is introduced.

In conclusion, we have shown that a set of expanding solutions exist in a model with two scalar fields coupled to the system. For the solutions to be expanding solutions, three independent inequalities (4.22) and (4.24) are required and listed clearly in this paper. These inequalities put constraints on the four parameters space spanned by λ_i and ρ_i . In particular, the inflationary solutions require a set of additional constraints on the four parameters space. These solutions are shown to be unstable.

As a brief summary, the cosmic no-hair conjecture asserts that all expanding solutions will tend to the de Sitter space asymptotically at time infinity. Formal and rigorous proof was shown in Ref. [5] showing that all expanding solutions will tend to de Sitter space asymptotically at time infinity for any model with a positive cosmological constant if both the SEC and DEC remain valid. A number of counterexamples are, however, shown to violate the SEC

and DEC [15–17]. As a result, these expanding solutions become a potential challenge to the cosmic no-hair conjecture.

Some of these solutions can be shown to be unstable by an exponential perturbation method with the perturbation functions proportional to $\exp[\nu t]$ [10,18–20]. The stability of some of these solutions are, however, still waiting for clarifications. Therefore, it is important to find out if there is any alternative and effective method to study the stability property of these expanding solutions.

Recently, a model with a scalar field coupled to the gauge field seems to provide another counterexample to the no-hair conjecture [59]. This model does not, however, have a cosmological constant. In addition, the power-law solutions of the scale factors expand in the form: $\exp[\alpha + \sigma] = t^{\zeta + \eta}$ and $\exp[\alpha - 2\sigma] = t^{\zeta - 2\eta}$ on the Bianchi type I space. These anisotropically expanding solutions appear to be stable. A new set of power-law perturbations is introduced to study the stability property of this set of expanding solutions. It can be shown, however, that inflationary solutions tend to be stable against the power-law perturbations.

In order to resolve the stability properties of the scalar-vector model, we introduced an additional phantom field to the one scalar model with the action given by Eq. (2.1). As a result, new exponential coupling with additional field parameters λ_2 and ρ_2 are introduced accordingly. A generalized set of anisotropically expanding solutions is also found to agree with the one scalar model [59] when we set $\rho_2 = 0$ and $\lambda_2 \rightarrow \infty$ together. If these solutions represent a set of expanding solutions, three

independent inequalities (4.22) and (4.24) are required to be observed. When power-law perturbations are applied to the field equations, we have shown that unstable mode does exist for inflationary solutions obeying the constraints $\rho_1 \lambda_2 + \rho_2 \lambda_1 \gg \lambda_1 \lambda_2 \sim O(1)$.

It is clear at this point that unstable mode does not exist for the one scalar model. And the presence of the phantom field perturbation introduces a change of sign to the leading coefficient b_8 of the polynomial equation $f(n) = 0$ representing the stability equation shown in Eq. (4.18) as compared to the leading coefficient c_6 of the polynomial equation $\tilde{f}(n) = 0$ in Eq. (5.3).

Apparently the induced sign flipping of the stability equation changes the stability property of the expanding solutions. In particular, we are able to show that the presence of the phantom field does drive the inflationary solutions off the anisotropically inflationary phase as discussed in this paper. The proof shown in this paper acts in favor of the cosmic no-hair conjecture. In addition, it is also a useful demonstration of a new approach of the stability analysis of relative models. Indeed, it is also consistent with the expectation that an additional direction of perturbations, the ψ field, for example, can introduce an additional unstable mode of the system. Hopefully the results shown in this paper can be helpful in a relative study on stability analysis.

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