A Derivation of the Glover–Doyle Algorithms for General $H^\infty$ Control Problems*

JANG-LEE HONG† and CHING-CHENG TENG†

Key Words—$H^\infty$ control; control algorithms; coprime factorization.

Abstract—We show that the Glover–Doyle algorithm can be formulated simply by using the $(J, J')$-lossless factorization method and chain scattering matrix description. This algorithm was first stated by Glover and Doyle in 1988. Because the corresponding diagonal block of the $(J, J')$-lossless matrix in the general 4-block $H^\infty$ control problem of the Glover–Doyle algorithm is not square, a new type of chain scattering matrix description is developed. With this description in hand, we obtain two types of state-space solution, which are similar to each other. Thus a similarity transformation between these solutions in the 4-block $H^\infty$ control problem can also be obtained. The main idea of the solution is illustrated by means of block diagrams.

1. Introduction

Since Zames (1981) proposed the concept of sensitivity minimization in the $H^\infty$ domain, many researchers have made valuable contributions to the study of the $H^\infty$ domain. Transparent controllers for the standard 4-block $H^\infty$ problem were not obtained until Glover and Doyle (1988, 1989) developed their well-known dual GD algorithms.

After Glover and Doyle (1989), Green et al. (1990) and Kimura (1991a) offered alternative developments using a J-spectral factorization, a characteristic of a $(J, J')$-lossless matrix. These methods are all based on the model-matching problem. Green (1992) combined an analytic system with J-lossless factorization to solve the $H^\infty$ control problem, which gradually yielded a problem in the form of the model-matching problem. Using $(J, J')$-lossless factorization and a chain-scattering matrix description, Kimura (1991b) and Ball et al. (1991) gave a fictitious signal method for solving the 4-block case of the problem. Furthermore, Kondo and Harz (1990) and Tsai and Tsai (1993) obtained results similar to those of Green (1992).

However, in these papers the (1,1) block or the (2,2) block of the $(J, J')$-lossless matrix is required to be square or to need additional fictitious signals. Consequently, the results obtained by using the $(J, J')$-lossless factorization method to solve the $H^\infty$ control problem were not the same as those obtained by using the Glover–Doyle algorithms. In this paper we combine a normalized coprime factorization of the plant and $(J, J')$-factorization of one of the coprime factors, together with an alternative type of chain matrix description to recover precisely the results of Glover and Doyle (1988) (by using a left-coprime factorization) and Glover and Doyle (1989) (by using a right coprime factorization).

Despite the specific features of the two cases, the transfer functions for the resulting compensators turn out to be the same. We also obtain an explicit state-space similarity between the realizations for the two compensators thus obtained.

In Section 2 we briefly state the standard $H^\infty$ control problem. The $(J, J')$-lossless, conjugate $(J, J')$-lossless and conjugate $(J, J')$-expansive matrices are also discussed. In Section 3 we develop alternative chain-scattering matrix descriptions, and discuss their chain properties. In Section 4 the relationship between the $H^\infty$ control problem and the chain scattering matrix description is stated. The main results and the solution are presented in Section 5.

2. Notation and preliminaries

Throughout this paper $\mathbb{R}$ denotes the real numbers, $RL^\infty$ denotes the set of proper real rational function matrices with no pole on the jw axis, and $RH^\infty$ denotes the $RL^\infty$ subspace with no poles in the right half-plane. Furthermore, $RH^\infty$ denotes the units of $RH^\infty$ (i.e. if $\Phi \in RH^\infty$ then $\Phi \in RH^\infty$ and $\Phi^{-1} \in RH^\infty$) and $BH^\infty := \{ \Phi \in RH^\infty \mid \| \Phi \| < \gamma \}$ denotes $G^\infty(\gamma)$, the set of Hamiltonian matrices with no pure imaginary eigenvalues, and $Rie(\gamma)$ is the unique solution of the corresponding ARE of the Hamiltonian matrix $H$. $G^\infty(\gamma)$ is equivalent to $\{ \Phi \in RH^\infty \mid \| \Phi \| < \gamma \}$.

The standard 4-block $H^\infty$ control problem. In the standard $H^\infty$ framework, the transfer functions from $u$ to $z$ are denoted by

$$ P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} A & B \\ C_1 & D_{11} \\ C_2 & D_{21} \end{bmatrix} $$

where $x(t) \in \mathbb{R}^n, y(t) \in \mathbb{R}^p, w(t) \in \mathbb{R}^m$, and $u(t) \in \mathbb{R}^q$ are the error, observation, disturbance and control input, respectively.

The suboptimal $H^\infty$ control problem is then modeled so as to choose a controller $K$, connecting the observation vector $y$ to $u$, such that $K$ internally stabilizes the closed-loop system. Furthermore, the closed-loop transfer function, denoted by

$$ F(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, $$

satisfies the $H^\infty$ norm bound

$$ \| F(P, K) \|_\infty < \gamma, \quad \gamma \in \mathbb{R}^+. $$

For simplicity and without loss of generality of the derivations in subsequent sections, we let $\| F(P_0, K) \|_\infty < \gamma$ instead of $\| F(P, K) \|_\infty < \gamma, i.e.$

$$ F(P, K) = \frac{1}{\gamma} F(P_0, K) = \frac{1}{\gamma} P_{11} + \frac{1}{\gamma} P_{12}K(I - P_{22}K)^{-1}P_{21}. $$

Figure 1 shows a general set-up for linear fractional transformation (LFT).

The assumption of the standard 4-block $H^\infty$ control problem are as follows.

* Received 24 March 1994; revised 23 February 1995; received in final form 4 September 1995. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor R. Tempo under the direction of Editor Ruth F. Curtain. Corresponding author Professor Ching-Cheng Teng. Fax: +886 (035) 715998.
† Institute of Control Engineering, National Chiao-Tung University, Hsinchu, Taiwan.

---

*Received 24 March 1994; revised 23 February 1995; received in final form 4 September 1995. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor R. Tempo under the direction of Editor Ruth F. Curtain. Corresponding author Professor Ching-Cheng Teng. Fax: +886 (035) 715998.
† Institute of Control Engineering, National Chiao-Tung University, Hsinchu, Taiwan.
Fig. 1. The general set-up for linear fractional transformation (LFT).

Assumptions.

A1. $(A, B)$ is stabilizable and $(C, A)$ is detectable.

A2. rank $D_{12} = m$, and rank $D_{21} = p$.

A3. (a) $D_{11}$ is stabilizable and $(C, A)$ is detectable.

A4. rank $D_{11} = m^2$ and rank $D_{12} = p^2$.

A5. rank $A - j \omega B = n + m_2$ for each $\omega \in \mathbb{R}$.

A6. rank $A - j \omega B = n + p_2$ for each $\omega \in \mathbb{R}$.

In the above assumptions, as in the general 4-block control problem, the inequalities $m_1 > p_2$ and $p_1 > m_2$ must hold.

Under the condition stated above, the main results of the Glover-Doyle algorithm are stated in Theorem 1 in Glover and Doyle (1988) and Theorem 4.1 in Glover and Doyle (1989).

2.2. $(J, J')$-lossless, conjugate $(J, J')$-lossless and conjugate $(J, J')$-expansive matrices.

A partitioned matrix $O(s) \in \mathbb{R}^{(r + e) \times (s + e)}$ is said to be $(J, J')$-lossless or $(J, J')$-lossless matrix if

$$\Theta(s) - \Theta(s)\Theta(s) = \Theta(s)$$

for each $s \in \mathbb{R}$, and

$$\Theta(s) - \Theta(s)\Theta(s) = \Theta(s)$$

for each $s \in \mathbb{R}$, where

$$J_{\omega} = \text{diag}(J_{\omega}, -I), \quad J_{\omega} = \text{diag}(J_{\omega}, -I).$$

Also, $\Theta(s)$ is called conjugate $(J_{\omega}, J_{\omega})$-lossless if (1) holds and $\Theta(s) \Theta(s) \leq J_{\omega}$ for each $s \in \mathbb{R}$. Finally, $\Theta(s)$ is called conjugate $(J_{\omega}, J_{\omega})$-expansive if (1) is satisfied and $\Theta(s) \Theta(s) \preceq J_{\omega}$ for each $s \in \mathbb{R}$.

Their relative properties are stated below. Here Lemma 1 is quoted from Kumura (1991a), and Lemmas 2 and 3 are extensions of Lemma 1 to stabilizable and observable realizations.

Lemma 1. Let $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(m + r) \times (p + q)}$, with $(A, B)$ controllable and $(C, A)$ detectable. Then $G$ is $(J_{\omega}, J_{\omega})$-lossless if

(i) $A^TX + XA + C^JT_{\omega}C = 0$;

(ii) $XB + C^JT_{\omega}D = 0$;

(iii) $B^TD_{\omega} = J_{\omega}$;

(iv) $X \geq 0$.

Lemma 2. Let $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(m + r) \times (p + q)}$, with $(A, B)$ observable and $(A, B)$ stabilizable. Then $G$ is conjugate $(J_{\omega}, J_{\omega})$-expansive if

(i) $AY = YA^T + B_{\omega}B^T = 0$;

(ii) $D_{\omega}B^T - CY = 0$;

(iii) $D_{\omega}D^T = J_{\omega}$.

(iv) $Y \geq 0$.

2.3. The $(J, J')$-lossless factorization. Since any real rational proper matrix $G(s)$ has a right- and a left-coprime factorization, we have $G = O \Theta^{-1} = O \Theta^{-1}$, where $O, \Pi, \Theta, \bar{O}, \bar{\Pi} \in \mathbb{R}^{H^+}$ and $\Pi(\bar{\Pi})$ are nonsingular.

We shall investigate below how to choose a particular state feedback gain matrix $F$, an observer gain matrix $H$, a scalar matrix $W$, and a scalar matrix $W$, such that $0$ is $(J_{\omega}, J_{\omega})$-lossless, $\bar{\Theta}$ is conjugate $(J, J')$-expansive and $\bar{\Pi} \in \mathbb{R}^{H^+}$ such that

$$W^*_e D^T J_{\omega} \bar{W}_e = J_{\omega}$$

Lemma 3. Let $G(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(m + r) \times (p + q)}$, with $(A, B)$ observable and $(A, B)$ stabilizable. Then $G$ is conjugate $(J_{\omega}, J_{\omega})$-lossless if

(i) $AY + YA^T + B_{\omega}B^T = 0$;

(ii) $D_{\omega}B^T = CY = 0$;

(iii) $D_{\omega}D^T = J_{\omega}$;

(iv) $Y \geq 0$.

Lemma 4. Let $G \in \mathbb{R}^{H^+ \times (m + r) \times (p + q)}$. Then there exists a right-coprime factorization (r.c.f.) $G = O \Theta^{-1}$ such that $\Theta$ is $(J_{\omega}, J_{\omega})$-lossless and $\bar{\Pi} \in \mathbb{R}^{H^+}$ if

(i) there exists a nonsingular matrix $W_e$ such that

$$W^*_e D^T J_{\omega} \bar{W}_e = J_{\omega}$$

(ii) $A_{e, 2} \in \text{dom}(\text{Ric})$ and $V = \text{Ric}(A_{e, 2}) \geq 0$, where

$$R_e = D^T J_{\omega} D$$

Lemma 5. Let $G \in \mathbb{R}^{H^+ \times (m + r) \times (p + q)}$. Then there exists a left-coprime factorization (l.c.f.) $G = O \Theta^{-1}$ such that $\bar{\Theta}$ is conjugate $(J_{\omega}, J_{\omega})$-expansive and $\bar{\Pi} \in \mathbb{R}^{H^+}$ if

(i) there exists a nonsingular matrix $W_e$ such that

$$W^*_e D^T J_{\omega} \bar{W}_e = J_{\omega}$$

(ii) $A_{e, 2} \in \text{dom}(\text{Ric})$ and $\bar{Z} = \text{Ric}(A_{e, 2}) \geq 0$, where

$$R_e = D^T J_{\omega} D$$

Lemma 6. Let $G \in \mathbb{R}^{H^+ \times (m + r) \times (p + q)}$. Then there exists a left-coprime factorization (l.c.f.) $G = O \Theta^{-1}$ such that $\bar{\Theta}$ is conjugate $(J_{\omega}, J_{\omega})$-expansive and $\bar{\Pi} \in \mathbb{R}^{H^+}$ if

$$W^*_e D^T J_{\omega} \bar{W}_e = J_{\omega}$$

Proof. This lemma can be obtained directly from Lemma 2.
changes when we combine these types of CSMD with the (J, J')-lossless property, the characteristics of these CSMDs are quite different from those of traditional CSMDs. Furthermore, the following various linear fractional transformations are defined only for those KS such that the inverse appearing in the formula exists.

Type I. If
\[ Q_{z1} \text{ is square}, \]
then
\[ z = \left( Q_{z1} + Q_{z2} \right) \left( Q_{z1} + Q_{z2} \right)^{-1} w. \]

Type II. If
\[ Q_{z1} \text{ is square}, \]
then
\[ w = \left( Q_{z1} + Q_{z2} \right) \left( Q_{z1} + Q_{z2} \right)^{-1} z. \]

Type III. If \( Q_{z1} \) is square, and
\[ \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -Q_{z2} \\ Q_{z1} & Q_{z2} \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \]
then
\[ w = (Q_{z1} - KQ_{z2})^{-1} (Q_{z1} - KQ_{z2}) w. \]

The superscript (1,2) indicates the location of the square matrix.

Type IV. If \( Q_{z2} \) is square, and
\[ \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & Q_{z2} \\ Q_{z1} & Q_{z2} \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}, \]
then
\[ z = (Q_{z1} - KQ_{z2})^{-1} (Q_{z1} - KQ_{z2}) w. \]

In the following lemmas, some properties of the above CSMDs (Types I–IV) are represented by the concept of an analytic system due to Green (1992), and are different from the traditional CSMDs. These properties are used to prove the sufficient condition of our main theorem.

Lemma 6. (Type I.) Assume that \( \Theta \) is a \((J_{m1}, J_{m2})\)-lossless matrix, in which \( Q_{z1} \) is square, and define
\[ F_{R}^{-1}(\Theta, K) = \Theta \Theta^* \]
Then \( F_{R}^{-1}(\Theta, K) = XY^{-1} \). Since \( \Theta \) is \((J_{m1}, J_{m2})\)-lossless, i.e., \( \Theta^* \Theta = I \), and
\[ \begin{bmatrix} X^* \\ Y^* \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \left[ \begin{array}{c} I \\ \Theta^* \Theta \end{array} \right] \left[ \begin{array}{c} \Phi \\ \Theta \end{array} \right], \]
we have
\[ X^*X + Y^*Y = I - \Phi \Phi^* \]
Thus
\[ \|\Phi\|_2 > 1 \Rightarrow \|F_{R}^{-1}(\Theta, K)\|_2 < 1. \]

Lemma 7. (Type III.) Assume that \( \Theta \) is a conjugate \((J_{m1}, J_{m2})\)-lossless matrix, in which \( Q_{z1} \) is square. Define
\[ F_{R}^{-1}(\Theta, K) = (Q_{z1} - Q_{z2})^{-1} (Q_{z1} - Q_{z2}) w. \]

Lemma 8. (Type IV.) Assume that \( \Theta \) is a conjugate \((J_{m1}, J_{m2})\)-lossless matrix, in which \( Q_{z2} \) is square. Define
\[ F_{R}^{-1}(\Theta, K) = (Q_{z1} - Q_{z2})^{-1} (Q_{z1} - Q_{z2}) w. \]

The following lemma proposed by Walker (1990) states the relationship between the solutions of two algebraic Riccati equations (ARE) whose Hamiltonian matrices are related by a similarity transformation. There is thus also a similarity transformation property between these solutions. Since this property enables us to simplify the derivation and gives us the similarity transformation of \( H^* \) controllers, we rewrite this lemma below.

Lemma 10. Let
\[ A_{H_r} = \begin{bmatrix} A_r & -R_r \\ -Q_r & -A_r^* \end{bmatrix} \in \text{dom (Ric)}, \]
and suppose that
\[ A_{H_r} = \begin{bmatrix} A_r & -R_r \\ -Q_r & -A_r^* \end{bmatrix} \]
is a Hamiltonian matrix given by
\[ A_{H_r} = T A_{H_r} T^{-1}, \]
where
\[ T = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}, \]
Thus, from (2), we have
\[ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_r - R_r \tilde{Z} \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -A_r - R_r \tilde{Z}^* \\ 0 \end{bmatrix} \]
\[ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_r - R_r \tilde{Z} \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -A_r - R_r \tilde{Z}^* \\ 0 \end{bmatrix}. \]


Furthermore, the above lemma entails that, if \( X \preceq 0, Y \preceq 0, I - XY > 0, I - YX > 0, \) and if there exists another Hamiltonian matrix \( A_{入} \), given by

\[ A_{入} = \begin{bmatrix}
-1 & -Y \\
0 & I
\end{bmatrix} A_{入} \begin{bmatrix}
0 & Y \\
1 & I
\end{bmatrix},
\]

with \( V = \text{Ric}(A_{入}), A_{入} \) is a Hamiltonian matrix and \( A_{入} \in \text{dom}(\text{Ric}), X = \text{Ric}(A_{入}), \) then

\[ \dot{Z} = Y(I - XY)^{-1}z, \]

\[ V = X(I - YX)^{-1}y, \]


As we shall state in Section 5, (7) is the similarity transformation in the 4-block \( H' \) controllers.

4. The relationship between GD algorithms and CSMD

Since our final objective in this paper is to derive the solutions in the distinct GD algorithms (Glover and Doyle 1988, 1989) simultaneously, and these solutions are related to the right- and left-coprime factorization of the augmented plant, we shall discuss both the right- and left-coprime cases.

4.1. Case I: right-coprime case. From Fig. 1, \( P = N M^{-1} \), and

\[ N = \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix}, \quad M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}, \quad N, M \in RH^{n}.
\]

We can obtain the structure shown in Fig. 2, with

\[ \begin{bmatrix}
z \\
y
\end{bmatrix} = \begin{bmatrix}
N_{11} & N_{12} \\
N_{21} & N_{22}
\end{bmatrix} \begin{bmatrix}
z' \\
w'
\end{bmatrix}, \quad \begin{bmatrix}
w \\
w'
\end{bmatrix} = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} \begin{bmatrix}
z' \\
w'
\end{bmatrix}.
\]

For convenience, we define \( G_{1} \) and \( G_{2} \) as

\[ \begin{bmatrix}
z \\
w
\end{bmatrix} = \begin{bmatrix}
N_{11} & N_{12} \\
M_{11} & M_{12}
\end{bmatrix} \begin{bmatrix}
z' \\
w'
\end{bmatrix}, \quad \begin{bmatrix}
w \\
w'
\end{bmatrix} = \begin{bmatrix}
M_{11} & M_{12} \\
N_{21} & N_{22}
\end{bmatrix} \begin{bmatrix}
z' \\
w'
\end{bmatrix}.
\]

and depict this in Fig. 3. Thus the state-space form of \( N, M, G_{1}, \) and \( G_{2} \) is

\[ \begin{bmatrix}
A + BF & BW_{e} \\
C + dF & DW_{e}
\end{bmatrix}, \]

where

\[ W_{e} = \begin{bmatrix}
W_{e_{11}} & W_{e_{12}} \\
W_{e_{21}} & W_{e_{22}}
\end{bmatrix}.
\]

Fig. 3. The chain-scattering matrix description (CSMD) of the right-coprime case.

is any nonsingular matrix. More precisely, we partition the matrices \( M \) and \( N \) in the forms

\[ M = \begin{bmatrix}
A + BF & BW_{e} \\
C + dF & DW_{e}
\end{bmatrix}, \quad N = \begin{bmatrix}
A + BF & BW_{e} \\
C + dF & DW_{e}
\end{bmatrix}, \]

\[ \begin{bmatrix}
C_{1} & D_{11} & F_{1} \\
C_{2} & D_{12} & F_{2}
\end{bmatrix}, \quad \begin{bmatrix}
C_{1} & D_{11} & F_{1} \\
C_{2} & D_{12} & F_{2}
\end{bmatrix}, \quad \begin{bmatrix}
C_{1} & D_{11} & F_{1} \\
C_{2} & D_{12} & F_{2}
\end{bmatrix},
\]

Then, from the preceding computations, we obtain \( G_{1} \) and \( G_{2} \) as

\[ G_{1} = \begin{bmatrix}
A + BF & BW_{e_{11}} + BW_{e_{12}} \\
C_{1} & D_{11} & F_{1}
\end{bmatrix}, \quad \begin{bmatrix}
A + BF & BW_{e_{11}} + BW_{e_{12}} \\
C_{2} & D_{12} & F_{2}
\end{bmatrix}
\]

where \( G_{1} \in RH^{n(1+m_{1}+m_{2})} \), and \( G_{2} \in RH^{n(1+m_{2}+m_{3})} \).

Remark 1. From Assumptions A1-A6, if we choose \( W_{e} \) as

\[ W_{e} = \begin{bmatrix}
(I - D_{11}^{T}D_{11})^{-1}D_{12}^{T}D_{12} & (I - D_{12}^{T}D_{12})^{-1}D_{12}^{T}D_{12} \\
0 & 0
\end{bmatrix}^{1/2}
\]

then \( G_{1} \) (9), will be \( (U_{p,m}, J_{m,m}) \)-lossless. Furthermore, if we rewrite \( G_{1} \) as

\[ G_{1}(s) = \begin{bmatrix}
A + BF & BW_{e} \\
C + dF & DW_{e}
\end{bmatrix}, \]

where \( A = A + BF \) and \( C = C + D, \) then, from Lemma 1,
we have the following properties (as in Glover et al., 1988, 1989):

(i) \( R = DT_1p_1D + D_t^T D_1 \)

(ii) \( XB + C^T JD = 0 \Rightarrow B^TX + RF + D^TJC = 0 \)

(iii) \( A^TX + XA + C^T(J - JD^TJC)C = 0 \)

From Section 1.4 in Glover and Doyle (1989), Assumptions A1 and A5 guarantee that the Hamiltonian matrix belongs to \( \text{dom}(Ric) \). So, fromLemma 4 in Doyle et al. (1989), if a Hamiltonian matrix \( H \) belongs to \( \text{dom}(Ric) \) then its solution \( Ric(H) \) exists, and thus the following solution exists:

\[
X = Ric\left(\begin{bmatrix} A & -BR^T \text{DTJC} & (A - BR^T \text{DTJC})^T \end{bmatrix}\right)
\]

where

\[
H = \begin{bmatrix} A & 0 & 0 \\ -CT_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

which is obtained by replacing (11) with (9).

4.2. Case II: left-coprime case. From Fig. 1 and \( P = M^*N \), with

\[
\hat{M} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{bmatrix},
\]

and using a similar procedure as in Fig. 2, we obtain

\[
\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}, \quad \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{bmatrix} \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}
\]

So

\[
\begin{bmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{bmatrix} \begin{bmatrix} \hat{A} \\ \hat{B} \end{bmatrix}
\]

Thus we have

\[
\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ \hat{N}_{21} & \hat{N}_{22} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}
\]

As stated in Remark 1, Assumptions A1 and A6 guarantee
that the Hamiltonian matrix belongs to Dom (Ric). Thus the following equation exists:

\[ Y = \text{Ric} \left( \begin{bmatrix} (A - BJDTk'C) & -C^T \tilde{R} & C \\ -B(J - JD^T R_k^T D)B^T & -B(J - JD^T R_k^T C) \end{bmatrix} \right) \geq 0 \]  

(16)

where

\[ J_k = \begin{bmatrix} A^T & 0 \\ -B_k D_k & -A \end{bmatrix} \begin{bmatrix} \tilde{R}^{-1} [D_k B_k^T & C] \end{bmatrix}, \]

which is obtained by substituting (14) into (16).

5. Main results

As a summary of the discussion so far, we state the following important theorems, which are the main tools we use to derive the Glover–Doyle algorithm. The two theorems both describe the results of the GD algorithm, but from different points of view.

Case I: the right-coprime case.


\[ P \in \mathbb{R}^{p \times p}_{+}, \quad p \times \pi_{m1} \]  

has the specific right-coprime factorization

\[ P = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{-1} \]

satisfying Remark 1. Then

(A) there exists an internally stabilizing controller \( K \) such that \( \|F(P, K)\|_\infty < 1 \) if

(i) \( G_1 := \begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix} \) is \((J_{m_{11}}, J_{m_1})\)-lossless,

(ii) \( G_2 := \begin{bmatrix} M_{12} & M_{22} \\ N_{12} & N_{22} \end{bmatrix} \) has a left-coprime factorization \( G_2 = \Phi \) such that \( \Phi \) is conjugate \((J_{m_{11}}, J_{m_1})\)-expansive and \( \Phi \notin \mathbb{R}^{m_{11}p \times p+}. \)

(B) if the conditions of (A) are satisfied then all real rational internally stabilizing controllers \( K \) such that \( \|F(P, K)\|_\infty < 1 \) are given by \( K = F_2(\hat{\Phi}, \Phi) \) \( \forall \Phi \in BH^* \).

Proof of necessity. From Section 4.1 (the selection of \( G_1 \) and \( G_2 \)) and Section 3 (Types I and III), we know that

\[ F(P, K) = F(NM^{-1}, K) \]

\[ = F_{1,2}(G_1, F_{1,2}(G_2, K)). \]

So, from Lemma 6, we have \( \|F(P, K)\|_\infty < 1 \) if \( G_1 \) is \((J_{m_{11}}, J_{m_1})\)-lossless and \( \|F_{1,2}(G_2, K)\|_\infty < 1 \). Since, from Remark 1, we have already obtained that \( G_1 \) is \((J_{m_{11}}, J_{m_1})\)-lossless, it remains to show that \( \|F_{1,2}(G_2, K)\|_\infty < 1 \).

By direct computation, we can verify that

\[ A_{H_2} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} J_2 \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, \]

(17)

which implies that \( A_{H_2} \) is similar to \( J_2 \), where \( X \) and \( J_2 \) are as shown in Remarks 1 and 2 and \( A_{H_2} \) is obtained by the following computation. Rewrite \( G_1 \) in (10) as

\[ G_2 = \begin{bmatrix} A_{G_2} & B_{G_2} \\ C_{G_2} & D_{G_2} \end{bmatrix}, \]

and factorize it as \( G_2 = \Phi \) such that

\[ \Phi = \begin{bmatrix} A_{G_2} + H_{G_2} C_{G_2} & H_2 & B_{G_2} + H_{G_2} D_{G_2} \\ W_1 C_{G_2} & W_2 & W_1 D_{G_2} \end{bmatrix} \]

(18)

and

\[ \Phi = \begin{bmatrix} C_{G_2} & A_{G_2} - B_{G_2} D_{G_2} R_{G_2} C_{G_2} \\ B_{G_2} (J - JD^T R_k D_k) B_k^T & -A_{G_2} - B_{G_2} D_{G_2} R_{G_2} C_{G_2} \end{bmatrix}, \]

(20)

\[ \Phi = \begin{bmatrix} C_{G_2} & A_{G_2} - B_{G_2} D_{G_2} R_{G_2} C_{G_2} \\ B_{G_2} (J - JD^T R_k D_k) B_k^T & -A_{G_2} - B_{G_2} D_{G_2} R_{G_2} C_{G_2} \end{bmatrix}. \]

Thus, from (5) in Lemma 10, with \( l - XY > 0 \), we can also have the solution of \( A_{H_2} \), i.e., \( \hat{Z} = \text{Ric}(A_{H_2}) \).

The above conditions show that \( G_2 = \hat{Z}^{-1} \hat{\Phi} \) satisfies Lemma 5, i.e. \( \hat{\Phi} \) is conjugate \((J_{m_{11}}, J_{m_1})\)-expansive and \( \hat{Z} \notin \mathbb{R}^{|m_{11}p | p+}. \)

5.2.2 Therefore, from Lemma 7, we have \( \|F_{1,2}(G_2, K)\|_\infty < 1 \). This completes the proof of necessity of (A).

Proof of sufficiency. Since

\[ F(P, K) = F_{1,2}(G_1, F_{1,2}(G_2, K)) \]

\[ = F_{1,2}(G_1, F_{1,2}(\hat{\Phi}, \Phi)), \]

\( \Phi \in BH^* \)

and \( G_1 \) is \((J_{m_{11}}, J_{m_1})\)-lossless, \( \hat{\Phi} \) is conjugate \((J_{m_{11}}, J_{m_1})\)-expansive. Thus, from Lemma 7, we have \( \|\Phi\|_\infty < 1 \Rightarrow \|F_{1,2}(\hat{\Phi}, \Phi)\|_\infty < 1 \). Furthermore, from Lemma 6, \( \|F_{1,2}(\hat{\Phi}, \Phi)\|_\infty < 1 \Rightarrow \|F_{1,2}(G_1, F_{1,2}(\hat{\Phi}, \Phi))\|_\infty < 1 \), so \( \|F(P, K)\|_\infty < 1 \).

The reason that \( K \) is an internally stabilizing controller is as follows. Let \( P_{22} \) have a doubly coprime factorization as

\[ P_{22} \rightarrow \tilde{k} \rightarrow \tilde{N} \rightarrow \tilde{N} \rightarrow \tilde{N} \rightarrow \tilde{k} \rightarrow \tilde{N} \rightarrow \tilde{N} \rightarrow I. \]

To see that the controller \( K \) is an internally stabilizing controller, let us consider the following computations. Redrawing Fig. 2 in Kondo and Hara (1990), we obtain Fig. 5. Rewriting (6.18) and (6.19) in Kondo and Hara (1990), we get

\[ \begin{bmatrix} T_2 & T_1 \\ 0 & T_0 \end{bmatrix} = \begin{bmatrix} P_{11} & T_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ M & \tilde{k} \tilde{N} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & \tilde{k} \tilde{N} \end{bmatrix}, \]

(21)

\[ \begin{bmatrix} T_2 & T_1 \\ 0 & T_0 \end{bmatrix} = \begin{bmatrix} \tilde{k} & \tilde{N} \\ -\tilde{N} & \tilde{k} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & \tilde{k} \tilde{N} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & \tilde{k} \tilde{N} \end{bmatrix}. \]

(22)

Figure 5 shows that the last term of each of the above equations will be cancelled in the closed-loop system. Hence, from Fig. 6, we see that the overall closed-loop system is constituted by the augmented plant and

\[ \begin{bmatrix} \tilde{k} & \tilde{N} \\ -\tilde{N} & \tilde{k} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & \tilde{k} \tilde{N} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & \tilde{k} \tilde{N} \end{bmatrix}. \]

Since we have not changed the structure of the augmented plant in our computation, \( K = F_2(\hat{\Phi}, \Phi) \) is thus equal to \( K = F_2(\hat{\Phi}, \Phi) \).

Fig. 5. The CSMD of model-matching problem for the 4-block H* control problem.
Q = RH* We conclude that K is an internally stabilizing controller, as in Doyle (1984). This completes the proof of sufficiency of (A).

Proof of (B). From the Youla parametrization, we know that all the internally stabilizing controllers can be represented in the form

\( K = FL(\Pi, \alpha) \quad \forall \alpha \in \mathcal{B}H^* \).

Theorem 1 can be described graphically as in Fig. 7.

In Section 5.1, we shall show that Case I leads to the same result as in Glover and Doyle (1989).

Case II: the left-coprime case.

**Theorem 2.** Under Assumptions A1–A6, suppose that \( P \in R\ell_{(p_1 + p_2) \times (m_1 + m_2)} \) has the specific left-coprime factorization

\[ P = \begin{bmatrix} \tilde{S}_{11} & \tilde{S}_{12} \\ \tilde{S}_{21} & \tilde{S}_{22} \end{bmatrix} \]

satisfying Remark 2. Then

(A) there exists an internally stabilizing controller K such that \( \| F(P, K) \|_\infty < 1 \) iff

(i) \( G_1 := \begin{bmatrix} \tilde{S}_{11} \\ \tilde{S}_{21} \end{bmatrix} \) is conjugate \((p_{21}, p_{22})\)-lossless,

(ii) \( \Psi := \begin{bmatrix} \tilde{S}_{22} & \tilde{S}_{21} \end{bmatrix} \) has a right-coprime factorization \( \Psi = \Theta \Pi^{-1} \) such that \( \Theta \) is \((p_{21}, p_{22})\)-lossless and \( \Pi \in \mathcal{B}H^{*}_{m_2 + m_2} \).

(B) if the conditions of (A) are satisfied then all the real rational internally stabilizing controllers K such that \( \| F(P, K) \|_\infty < 1 \) are given by \( K = F_\Theta \Pi \) \( \forall \Theta \in \mathcal{B}H^* \).

Proof. This follows by the same lines as for the right-coprime case.

Note that, by a similar computation to that in Theorem 1, we have the following properties:

\[ A_{K,1} = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} A_{H,1} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \] (23)

where \( A_{H,1} \) and \( H \) are shown in Remarks 1 and 7, and \( A_{K,1} \) is obtained as follows. Rewrite \( \Psi \) in (15) as

\[ \Psi = \begin{bmatrix} A_{\Psi,1} & B_{\Psi,1} \\ C_{\Psi,1} & D_{\Psi,1} \end{bmatrix} \]

and let \( \Psi = \Theta \Pi^{-1} \), where

\[ \begin{bmatrix} \Pi \\ \Theta \end{bmatrix} = \begin{bmatrix} A_{\Psi,1} + B_{\Psi,1} F_2 & B_{\Psi,1} W_2 \\ C_{\Psi,1} + D_{\Psi,1} F_2 & D_{\Psi,1} W_2 \end{bmatrix} \] (24)

and

(i) \( W_2 D_2^T J D_2 W_2 - J \); (25)

(ii) \( R_2 = D_2^T J D_2 \).

(iii) \( A_{H,2} = \text{dom} \left( R \right) \) and \( V = \text{Ric} \left( A_{H,2} \right) \) as we obtain in (6).

\[ A_{H,2} = \begin{bmatrix} A_{\Psi,1} - B_{\Psi,1} R_2^{-1} J D_2 \Psi W_2 & C_2 (J - D_2 R_2^{-1} D_2) C_2 \\ -C_2 (J - D_2 R_2^{-1} D_2) C_2 & A_{\Psi,1} - B_{\Psi,1} R_2^{-1} J D_2 \Psi W_2 \end{bmatrix} \] (26)

(iv) \( F_2 = -R_2^{-1} (D_2^T V + D_2^T J C_2 W_2) \).

Theorem 2 can be illustrated as shown in Fig. 8.

In Section 5.1, we shall show that Case II leads to the same result as in Glover and Doyle (1988).

5.1. The derivation of the controller K.. In this subsection we show how to derive the controllers \( K_{\alpha} \) of the GD algorithm by the relationship between \( \Pi \) and \( \Pi_1 \). Furthermore, we state the similarity transformation of these solutions; this also implies that the \( K_{\alpha} \)s in Glover and Doyle (1988,1989) are the same. As we shall discuss, the controllers \( K_{\alpha} \) can be found directly from the relationship between the structure of CSMD and the linear fractional transformation (LFT).

First, in the right-coprime case, where, from (10) and (18), we have

\[ A_{C,2} = A + BF, \quad H_{2} = [H_{1z}, H_{2z}], \quad C_{C,2} = F_2 \begin{bmatrix} C_{2} + D_{21} F_1 \end{bmatrix}, \]

\[ \Pi = \begin{bmatrix} A + BF + H_{1z} F_2 + H_{2z} (C_{2} + D_{21} F_1) & H_{1z} & H_{2z} \\ F_2 & W_2 & C_{2} + D_{21} F_1 & \end{bmatrix} \]

(27)

Similarly, in the left-coprime case, from (15) and (24), because

\[ A_{\Psi} = A + HC, \quad B_{\Psi} = [B_{2} + H_{1} D_{12}, H_{2}], \quad F_{\Psi} = \begin{bmatrix} F_2 \\ F_{\alpha} \end{bmatrix} \]

we have

\[ \pi_1 = \begin{bmatrix} A + HC + (B_{1} + H_{1} D_{12}) F_{\alpha} + H_{2} F_{\alpha} & [B_{2} + H_{1} D_{12}, H_{2}] W_2 \end{bmatrix} \]

\[ \begin{bmatrix} F_1 & -F_2 \end{bmatrix} \]

(28)

If we use Lemma 10 and substitute (17) and (20) into (4), we obtain

\[ (I + \hat{Z} X) (A + HC) = (A_{\alpha} + H C_{\alpha}) (I + \hat{Z} X) \]

\[ = (A + BF + H_{1z} \begin{bmatrix} F_2 \\ C_{2} + D_{21} F_1 \end{bmatrix}) (I + \hat{Z} X) \].

(29)
Substituting (23) and (26) into (8), we obtain

(30)

Note that in (7), \( I - \hat{X} = I + YV = (I - YX)^{-1} \). Thus, if we let \( Z = I + \hat{X} = I + YV = (I - YX)^{-1} \) and substitute (29) into (30), we obtain

\[H^\dagger \begin{bmatrix} F_2 \\ C_2 + D_{21}F_1 \end{bmatrix} Z = -Z[B_2 + H_1D_{12} - H_2F_2] \]
(31)

One of the solutions of (31) is

\[H_1 = -Z[B_2 + H_1D_{12} - H_2F_2] \]
(32)

where \( J = \text{diag}(I, -I) \). Therefore, for the right-coprime case, if we let

\[W_e = \begin{bmatrix} \hat{D}_{12} & -\hat{D}_{12} \hat{D}_{11} \end{bmatrix} \]
(33)

satisfying (19), where \( W_e \) can be obtained by properly choosing \( D_{11}, \hat{D}_{12} \) and \( D_3 \), such that

\[ \hat{D}_{12}D_{12} - \hat{D}_{12} \hat{D}_{11} = [D_{12}(I - D_1D_1^\dagger)D_{12}]^{-1}, \]
\[ \hat{D}_{12} \hat{D}_{11} = I, \]
then (27) becomes

\[\hat{F} = \begin{bmatrix} A + BF + H_1F_2 + H_2(C_2 + D_{21}F_1) \\ \hat{D}_{12}^2 + \hat{D}_{12} \hat{D}_{11} \end{bmatrix} \]
(34)

The internally stabilizing controller in CSMD form is \( F_e(\Pi, \Phi) \). This needs to be transformed into LFT form as shown in Fig. 9. If we rewrite \( \Pi \) as

\[\hat{\Pi} = \begin{bmatrix} \hat{\Pi}_{n1} & \hat{\Pi}_{n2} \\ \hat{\Pi}_{n1} & \hat{\Pi}_{n2} \end{bmatrix} \begin{bmatrix} A_0 & B_{10} & B_{20} \\ C_{10} & D_{10} & D_{20} \end{bmatrix} \]
(35)

then, from Fig. 9, we can see that the CSMD form is

\[\begin{bmatrix} u \\ \sigma \end{bmatrix} = \begin{bmatrix} \hat{\Pi}_{n1} & \hat{\Pi}_{n2} \\ \hat{\Pi}_{n1} & \hat{\Pi}_{n2} \end{bmatrix} \begin{bmatrix} u \\ \sigma \end{bmatrix} \]
(36)

and the LFT form is

\[\begin{bmatrix} u \\ \sigma \end{bmatrix} = \begin{bmatrix} -\hat{\Pi}_{n1} & \hat{\Pi}_{n2} \\ \hat{\Pi}_{n1} & \hat{\Pi}_{n2} \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix} = K_e \begin{bmatrix} y \\ v \end{bmatrix} \]
(37)

Therefore, we have the following state-space representation of \( K_e \):

\[K_e = \begin{bmatrix} A_0 & B_{10} & B_{20} \\ C_{10} & D_{10} & D_{20} \\ C_{10} & D_{10} & D_{20} \end{bmatrix} \begin{bmatrix} D_{10}^{-1} \hat{D}_{10} & -D_{10}^{-1} \hat{D}_{10}D_{20} & -D_{10}^{-1} \hat{D}_{10} \hat{D}_{20} \\ -D_{10}^{-1} \hat{D}_{10} & -D_{10}^{-1} \hat{D}_{10}D_{10} & -D_{10}^{-1} \hat{D}_{10} \hat{D}_{10} \\ -D_{10}^{-1} \hat{D}_{10} & -D_{10}^{-1} \hat{D}_{10}D_{10} & -D_{10}^{-1} \hat{D}_{10} \hat{D}_{10} \end{bmatrix} \]
(38)

Now we use the same notation as in the Glover-Doyle algorithm, i.e.

\[K_e = \begin{bmatrix} A_0 & B_{10} & B_{20} \\ C_{10} & D_{10} & D_{20} \\ C_{10} & D_{10} & D_{20} \end{bmatrix} \begin{bmatrix} \hat{D}_{10} & \hat{D}_{10} \hat{D}_{20} \\ -\hat{D}_{10} & \hat{D}_{10} \hat{D}_{20} \end{bmatrix} \]
(39)

and, replacing (36) by (34), we obtain

\[A = A + BF + \hat{D}_{10} \hat{D}_{20} \hat{D}_{21}, \]
\[B_1 = -ZH_1 + \hat{D}_{10} \hat{D}_{20} \hat{D}_{11}, \]
\[B_2 = ZH_2 + \hat{D}_{10} \hat{D}_{20} \hat{D}_{12}, \]
\[C_1 = \hat{D}_{10} \hat{D}_{20} \hat{D}_{21} \hat{D}_{22}, \]
\[C_2 = \hat{D}_{10} \hat{D}_{20} \hat{D}_{22}, \]
(40)

which are the same as in Theorem 4.1 in Glover and Doyle (1989).

Furthermore, for the left-coprime case, if

\[W_e = \begin{bmatrix} \hat{D}_{12} & 0 \\ 0 & -\hat{D}_{12} \end{bmatrix} \]
(41)

satisfies

\[\hat{D}_{12} \hat{D}_{21} = [D_{12}(I - D_1D_1^\dagger)D_{12}]^{-1}, \]
then \( W_e \) satisfies (25). Thus from (28), we have

\[\begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} 0 & \hat{D}_{12} \hat{D}_{21} \\ 0 & \hat{D}_{12} \hat{D}_{21} \end{bmatrix} \]
(42)

We also have \( K(s) = F_e(\Pi, \Phi) \). Transforming this to LFT form, graphically, we obtain the result shown in Fig. 10, and

\[K_e = \begin{bmatrix} A_0 & B_{10} & B_{20} \\ C_{10} & D_{10} & D_{20} \\ C_{10} & D_{10} & D_{20} \end{bmatrix} \begin{bmatrix} D_{10}^{-1} \hat{D}_{10} & -D_{10}^{-1} \hat{D}_{10}D_{20} & -D_{10}^{-1} \hat{D}_{10} \hat{D}_{20} \\ -D_{10}^{-1} \hat{D}_{10} & -D_{10}^{-1} \hat{D}_{10}D_{10} & -D_{10}^{-1} \hat{D}_{10} \hat{D}_{10} \\ -D_{10}^{-1} \hat{D}_{10} & -D_{10}^{-1} \hat{D}_{10}D_{10} & -D_{10}^{-1} \hat{D}_{10} \hat{D}_{10} \end{bmatrix} \]
(43)
Fig. 10. The transformation of the internally stabilizing controller from right CSMD to LFT.

Using the same notation as in the GD algorithm and substituting (38) into (39), we obtain

\[\dot{A} = A + HC + \dot{B}zD_{12}\dot{C}_1,\]

\[\dot{B}_1 = -H_2 + \dot{B}_2D_{12}D_{11},\]

\[\dot{B}_2 = (B_2 + H_1D_{12})D_{11},\]

\[\dot{C}_1 = F_zZ + F_tN_1F_2\dot{C}_2,\]

\[\dot{C}_2 = -D_{22}(C_2 + F_tN_1)^T,\]

which is equivalent to Theorem 1 in Glover and Doyle (1988).

5.2. The similarity transformation of the dual solutions. We know that the transfer function of the dual solutions are equivalent. Thus there must exist a similarity transformation between these dual state-space solutions.

If we substitute (32) into (29), we have

\[Z(A + HC) = (A, + H,D,C,-,+ \dot{C}_2)Z\]

\[= (A + BF)Z - Z(H_2 + H,D,C,-,+ \dot{C}_2)Z + ZH_2(C_2 + D_2F_0)Z.\]

(41)

Substituting (33) into (30), we obtain

\[(A + BF)Z = Z(A_\infty + B_uF_u)\]

\[= Z(A + HC) + Z(B_2 + H_1D_1)F_2Z\]

\[= ZH_2(C_2 + D_2F_0)Z.\]

(42)

Thus, if we compare (37) and (40) with (41) and (42), we find that the similarity transformation between the controllers of the dual case is \(Z = (I - YY)^{-1}\). Using subscripts 1988 and 1989 to denote the results in Glover and Doyle (1988, 1989), we have

\[
\begin{bmatrix}
A_{1988} & B_{1988} \\
C_{1988} & D_{1988}
\end{bmatrix}
= \begin{bmatrix}
Z A_{1988} Z^{-1} & Z B_{1988} \\
Z C_{1988} Z^{-1} & Z D_{1988}
\end{bmatrix}
= \begin{bmatrix}
A_{1989} & B_{1989} \\
C_{1989} & D_{1989}
\end{bmatrix}
\]

This relationship also means that the \(K_\infty\)'s in Glover and Doyle (1988, 1989) are the same.

6. Conclusions

We have combined coprime factorization and \((J, J')\)-lossless factorization to derive the two distinct Glover-Doyle algorithms of Glover and Doyle (1988, 1989). We have also stated sufficient and necessary conditions for the existence of all controllers \(K(t)\). Because the corresponding square matrix of the \((J, J')\)-lossless matrix in the Glover-Doyle algorithms is not on the diagonal block, some alternative chain scattering matrix descriptions have been proposed. Furthermore, a similarity transformation between these standard 4-block \(H^\infty\) controllers has been given.

Acknowledgements—The authors would like to thank Professor C. A. Lin of Chiao-Tung University for numerous helpful discussions and for carefully reviewing the original manuscript of this paper. We also thank the referees for their constructive comments and suggestions. This work was supported by the National Science Council, R.O.C., under Contract NSC 83-0424-E-009-005.

References


