Panpositionable Hamiltonicity of the Alternating Group Graphs

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The alternating group graph $AG_n$ is an interconnection network topology based on the Cayley graph of the alternating group. There are some interesting results concerning the hamiltonicity and the fault tolerant hamiltonicity of the alternating group graphs. In this article, we propose a new concept called panpositionable hamiltonicity. A hamiltonian graph $G$ is panpositionable if for any two different vertices $x$ and $y$ of $G$ and for any integer $l$ satisfying $d(x, y) \leq l \leq |V(G)| - d(x, y)$, there exists a hamiltonian cycle $C$ of $G$ such that the relative distance between $x$, $y$ on $C$ is $l$. We show that $AG_n$ is panpositionable hamiltonian if $n \geq 3$. © 2007 Wiley Periodicals, Inc.

NETWORKS, Vol. 50(2), 146–156 2007
DOI 10.1002/net

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1. INTRODUCTION

Network topology is usually represented by a graph where the vertices represent processors and the edges represent the links between processors. For graph definitions and notation we follow Ref. [6]. Let $G = (V, E)$ be a graph, where $V$ is a finite set and $E$ is a subset of $\{(u, v) \mid (u, v)$ is an unordered pair of $V\}$. We say that $V$ is the vertex set and $E$ is the edge set of $G$. Two vertices $u$ and $v$ are adjacent if $(u, v) \in E$. A path $P$ is represented by $\langle v_0, v_1, v_2, \ldots, v_k \rangle$. The length of a path $P$ is the number of edges in $P$, denoted by $L(P)$. We sometimes write the path $\langle v_0, v_1, v_2, \ldots, v_k \rangle$ as $\langle v_0, P_1, v_1, v_{i+1}, \ldots, v_j, P_2, v_i, \ldots, v_k \rangle$, where $P_1$ is the path $\langle v_0, v_1, \ldots, v_i \rangle$ and $P_2$ is the path $\langle v_j, v_{j+1}, \ldots, v_k \rangle$. It is possible to write a path $\langle v_0, v_1, P, v_1, v_2, \ldots, v_k \rangle$ if $L(P) = 0$. We use $d_C(u, v)$, or simply $d(u, v)$ if there is no ambiguity, to denote the distance between $u$ and $v$ in a graph $G$, i.e., the length of a shortest path joining $u$ and $v$ in $G$. A cycle is a path with at least three vertices such that the first vertex is the same as the last one. We use $d_C(u, v)$ and $D_C(u, v)$ to denote the shorter and the longer distance between $u$ and $v$ on a cycle $C$ of $G$, respectively. It is possible that $D_C(u, v) = d_C(u, v)$ if the lengths of the two disjoint paths joining $u$ and $v$ in $C$ are equal. A path is a hamiltonian path if its vertices are distinct and span $V$. A graph $G$ is hamiltonian connected if there exists a hamiltonian path joining any two vertices of $G$. A hamiltonian cycle of $G$ is a cycle that traverses every vertex of $G$ exactly once. A graph $G$ is hamiltonian if there exists a hamiltonian cycle in $G$. The hamiltonian properties are important aspects of designing an interconnection network. Many related works have appeared in the literature [1, 5, 7].

We propose a new concept called panpositionable hamiltonicity. A hamiltonian graph $G$ is panpositionable if for any two different vertices $x$ and $y$ of $G$ and for any integer $l$ satisfying $d(x, y) \leq l \leq |V(G)| - d(x, y)$, there exists a hamiltonian cycle $C$ of $G$ such that the relative distance between $x$, $y$ on $C$ is $l$; more precisely, $d_C(x, y) = l$ if $l \leq \lfloor\frac{|V(G)|}{2}\rfloor$ or $D_C(x, y) = l$ if $l > \lfloor\frac{|V(G)|}{2}\rfloor$. Given a hamiltonian cycle $C$, if $d_C(x, y) = l$, we have $D_C(x, y) = |V(G)| - d_C(x, y)$. Therefore, a graph is panpositionable hamiltonian if for any integer $l$ with $d(x, y) \leq l \leq \lfloor\frac{|V(G)|}{2}\rfloor$, there exists a hamiltonian cycle $C$ of $G$ with $d_C(x, y) = l$. One trivial example, the complete graph $K_n$ with $n \geq 3$, is panpositionable.

There are several requirements in designing a good topology for an interconnection network, such as connectivity and hamiltonicity. The hamiltonian property is one of the major requirements in designing an interconnection network. The hamiltonian property is fundamental to the deadlock-free routing algorithms of distributed systems [8, 9]. A high-reliability network design can be based on constructing a hamiltonian cycle in an interconnection network. Similar
to the importance of hamiltonicity for the communication between processors in an interconnection network, panpositionable hamiltonicity allows more flexible communication in a hamiltonian network. The panpositionable hamiltonian property inherits the hamiltonian property and advances it further. The concept is interesting and useful in the study of interconnection networks.

The alternating group graph [7] was proposed by Jwo et al. The interconnection scheme is based on the Cayley graph of the set of all even permutations. Cheng and Lipman investigated some properties of this family of graphs in Refs. [2–4]. Let \( n \) be \( \{1, 2, \ldots, n\} \). Then a permutation of elements in \( n \) is a permutation if \( p_i \neq p_j \) for all \( i \neq j \). Suppose that \( p_i \) and \( p_j \) are two different symbols in \( p \) with \( i < j \); then the pair \((i, j)\) is an inversion if \( p_i > p_j \). A permutation is even if the number of inversions is even. Let \( g_i^+ \) and \( g_i^- \) be two operations such that \( pg_i^+ \) and \( pg_i^- \) are permutations obtained from permutation \( p \) by rotating the symbols in position \( 1, 2, \ldots, n \), and \( p \) from left to right and from right to left, respectively.

For example, we have \( 1234g_{1}^{+} = 4132 \) and \( 1234g_{1}^{-} = 2431 \). The \( n \)-dimensional alternating group graph \( AG_n \) is the graph \((V_n, E_n)\), where \( V_n \) is the set of all even permutations, and \( E_n = \{(p, q) \mid p, q \in V_n, \text{ and } p = g_i^+ q \text{ or } q = g_i^- p \text{ for } i = 3, 4, \ldots, n\} \). Figure 1 illustrates \( AG_3 \) and \( AG_4 \). An alternating group graph \( AG_n \) is a regular graph of degree \( 2(n - 2) \) with \( n! \) vertices, and \( AG_n \) is vertex symmetric and edge symmetric [7]. The diameter of \( AG_n \) is \( [\frac{3n(n - 2)}{2}] \). There are some studies concerning hamiltonicity of the alternating group graph. Jwo et al. [7] showed that the alternating group graph is hamiltonian and hamiltonian connected. Chang et al. [1] showed that \( AG_n \) is \((n - 2)\)-vertex fault tolerant hamiltonian and \((n - 3)\)-vertex fault tolerant hamiltonian connected if \( n \geq 4 \). A graph \( G \) is panconnected if there exists a path of length \( l \) joining any two different vertices \( x \) and \( y \) with \( d(x, y) \leq l \leq |V(G)| - 1 \). A graph \( G \) is pancyclic if it contains a cycle of length \( l \) for each \( l \) satisfying \( 3 \leq l \leq |V(G)| \). Chang et al. also proved that \( AG_n \) is panconnected and pancyclic for all \( n \geq 3 \) [1].

In this article, we study the panpositionable hamiltonicity of the alternating group graph \( AG_n \). We show that the alternating group graph \( AG_n \) is panpositionable hamiltonian for all \( n \geq 3 \). In the following section, we discuss some basic properties of the alternating group graph. In Section 3, we prove our main theorem. In the final section, we present our conclusion and explain some relationship between the panpositionable hamiltonian property and the panconnected property.

2. SOME PROPERTIES OF THE ALTERNATING GROUP GRAPHS

For each \( n \geq 3 \), let \( V_n[i] = \{p \mid p = p_1p_2 \ldots p_n \text{ and } p_k = i\} \). It is the set of all vertices with the rightmost position \( i \). Let \( AG_n[i] \) denote the subgraph of \( AG_n \) induced by \( V_n[i] \). It is easy to see that each \( AG_n[i] \) is isomorphic to \( AG_{n-1} \). Thus, \( AG_n \) can be recursively constructed from \( n \) copies of \( AG_{n-1} \). Each \( AG_n[i] \) represents a subcomponent of \( AG_n \). Let \( I \) be a subset of \( \{1, 2, \ldots, n\} \). We use \( AG_n(I) \) to denote the subgraph of \( AG_n \) induced by \( \bigcup_{i \in I} V_n[i] \). We call \( AG_n(I) \) an incomplete alternating group graph if \( |I| < n \). We observe that each \( AG_n[i] \) can be recursively decomposed into its smaller subcomponents. We use \( E_n[I] \) to denote the set of edges between \( AG_n[I] \) and \( AG_n[j] \).

Proposition 1. Let \( n \) be an integer with \( n \geq 5 \), and let \( i \) and \( j \) be two distinct elements of \( \{n\} \). Suppose that \( H \) is one subcomponent of \( AG_n[I] \) with the \((n - 1)\)th position being \( h \) and the \( n \)th position being \( j \) for some \( h \in \{n \} - \{i, j\} \). Then \( |E_n[I]| = (n-2)! \), and the number of edges between \( AG_n[I] \) and \( H \) is \(|n-3|! \). Moreover, if \((u, v) \) and \((u', v') \) are distinct edges in \( E_n[I] \), then \( u, v, (u, v) \cap F = \emptyset \), and \((u, u') \in E(AG_n[I]) \) if and only if \((v, v') \in E(AG_n[I]) \).

Let \( i \in \{n\} \), and let \( u \) be a vertex in \( AG_n[I] \). We say that \( v \) is a neighbor of \( u \) if \( v \) is adjacent to \( u \). We use \( N^+(u) \) to denote the set of all neighbors of \( u \), which are in \( AG_n(I) \). Particularly, we use \( N^+(u) \) as an abbreviation of \( N^+(u) - \{i, j\} \). We call vertices in \( N^+(u) \) the outer neighbors of \( u \). Obviously, \(|N^+(u)| = 2(n - 3) \) and \( |N^+(u)| = 2 \). We say that vertex \( u \) is adjacent to subcomponent \( AG_n[I] \) if \( u \) has an outer neighbor in \( AG_n[I] \). Then, we define the adjacent subcomponent \( AS(u) \) of \( u \) as \( [i \mid u \text{ is adjacent to } AG_n[I]\} \). We have the following proposition:

Proposition 2. Suppose that \( n \geq 4 \) and \( i \in \{n\} \). Let \( u \) and \( v \) be two distinct vertices in \( AG_n[I] \).

(a) If \( d(u, v) = 1 \), then \( |AS(u) \cap AS(v)| = 1 \) and \( AS(u) \neq AS(v) \).

(b) If \( d(u, v) = 2 \), then \( AS(u) \neq AS(v) \).

Proof. Let \( u = u_1u_2 \ldots u_n, v = v_1v_2 \ldots v_n, \) and \( u_n = v_n = i \). If \( d(u, v) = 1 \), we have \( v = u_ig_k \) for some \( 3 \leq k \leq n - 1 \) and \( c \in \{+, -\} \). Without loss of generality, we may assume that \( c = + \). This means that \( u_1 = v_2, u_2 = v_k, \)
and $u_k = v_1$ if $v = u_k^+$. Let $u_k^+$ and $u_k^-$ be the two outer neighbors of $u$. Let $v_k^+$ and $v_k^-$ be the two outer neighbors of $v$. Then, $AS(u) = \{u_1, u_2\}$ and $AS(v) = \{v_1, v_2\}$. Thus $AS(u) \cap AS(v) = \{u_1\} = \{v_2\}$, and $|AS(u) \cap AS(v)| = 1$. It is obvious that $AS(u) \neq AS(v)$.

If $d(u, v) = 2$, there exists a vertex $w \in V(AG_n[i])$ such that $d(u, w) = d(v, w) = 1$, because $u, v \in AG_n[i]$. Let $w = w_1w_2 \cdots w_n$. If $u = v = v_1 = v_2 = v_3 = v_4$, then $AS(u) = \{w_1, w_2\}$ and $AS(v) = \{w_2, w_4\}$. If $u = v = v_1 = v_2 = v_3 = v_4$, then $AS(u) = \{w_1, w_3\}$ and $AS(v) = \{w_2, w_4\}$. Thus $d(u, v) = 2$. Hence the statement follows.

Chang et al. studied the fault hamiltonicity and fault hamiltonian connectivity of the alternating group graphs in Ref. [1]. The result is listed as follows.

**Theorem 1.** [1] Alternating graphs $AG_n$, $n \geq 4$, are $n - 2$ vertex-fault tolerant hamiltonian and $n - 3$ vertex-fault tolerant hamiltonian connected.

The above theorem states that with up to $n - 2$ faulty vertices $AG_n$ still has a hamiltonian cycle, and with up to $n - 3$ faulty vertices $AG_n$ is still hamiltonian connected. The following lemmas consider the hamiltonian connectivity of the subgraphs $AG_n(I)$ of the alternating group graphs $AG_n$.

**Lemma 1.** Suppose that $I \subseteq \{1, 2, 3, 4\}$ with $|I| \geq 2$. If $x \in V(AG_n[I])$ and $y \in V(AG_n[I])$, then there is an edge between $AG_4[I]$ and $AG_4[I]$ with $i \neq j \in I$, then there is a hamiltonian path of $AG_4[I]$ joining $x$ and $y$.

**Proof.** The alternating group graph $AG_4$ is known to be edge symmetric. Without loss of generality, we may consider that $x = 3241$ and $y = 1342$, which are adjacent vertices of $AG_4$ in Figure 1. If $I = \{1, 2\}$, then $\{3241, 4321, 2431, 4132, 3412, 1342\}$ forms a hamiltonian path of $AG_4(I)$ from $x$ to $y$. If $I = \{1, 2, 3\}$, then $\{3241, 4321, 4231, 2432, 1423, 2143, 4123, 3412, 1342\}$ forms a hamiltonian path of $AG_4(I)$ from $x$ to $y$. If $I = \{1, 2, 3, 4\}$, then by Theorem 1, $AG_4$ is hamiltonian connected. Hence the lemma follows.

**Lemma 2.** Suppose that

1. $n \geq 5$,
2. $I \subseteq \{n\}$ with $|I| \geq 2$,
3. $F \subseteq V(AG_n) \cup E(AG_n)$, and
4. $AG_n[I]$ is a hamiltonian connected for each $I \in I$ and $|F| \leq 2n - 7$.

Then, for any $x \in V(AG_n[I])$ and $y \in V(AG_n[I])$ with $i \neq j \in I$, there is a hamiltonian path of $AG_n(I - F)$ joining $x$ and $y$.

**Proof.** Consider that $|F| \leq 2n - 7$. Suppose that $|GE_n[I,F]| < 3$ for some $i_1, i_2 \in \{n\}$. Since $|GE_n[I,F]| = (n - 2)! \geq 2(n - 2)$, this implies that $|F| > 2n - 7$. We get a contradiction. Hence we have $|GE_n[I,F]| \geq 3$. We prove this lemma by induction on $|I|$. Suppose that $|I| = 2$, and $I = \{i, j\}$ for some $i, j$. Since $|GE_n[I,F]| \geq 3$, there exists an edge $(u, v) \in GE_n(I,F)$ such that $u \neq x \in V(AG_n[I])$ and $v \neq y \in V(AG_n[I])$. By the assumption that each $AG_n[I - F]$ is hamiltonian connected, there is a hamiltonian path $P_1$ of $AG_n[I - F]$ from $x$ to $u$ and a hamiltonian path $P_2$ of $AG_n[I - F]$ from $v$ to $y$. Thus $(x, P_1, u, v, P_2, y)$ forms a hamiltonian path of $AG_n[I] - F$ from $x$ to $y$.

Assume that the result is true for all $I'$ with $2 \leq |I'| < |I|$. Thus there is an $I' \subseteq I$ with $i \neq j$. Since $|GE_n[I',F]| \geq 3$, there exists an edge $(u, v) \in GE_n[I',F]$ such that $u \neq x \in V(AG_n[I'])$ and $v \neq y \in V(AG_n[I'])$. Then, there is a hamiltonian path $P_1$ of $AG_n[I - \{j\}] - F$ from $x$ to $u$ and a hamiltonian path $P_2$ of $AG_n[I - \{j\}] - F$ from $v$ to $y$. Thus $(x, P_1, u, v, P_2, y)$ forms a hamiltonian path of $AG_n[I] - F$ from $x$ to $y$. Hence the lemma follows.

Jwo et al. [7] presented a shortest path routing algorithm for the alternating graph group $AG_n$, and gave some characterizations of the minimum length path between two arbitrary vertices in $AG_n$. With this algorithm, we can find a minimum length path between any two distinct vertices of $AG_n$ as stated in the following proposition.

**Proposition 3.** [7] Let $i, j \in \{n\}$ and $i \neq j$. Suppose that $u = u_1u_2 \cdots u_n$ and $v = v_1v_2 \cdots v_n$ are two vertices in $AG_n$.

(a) If $u_n = v_n = i$, then $u$ and $v$ belong to the same subcomponent $AG_n[I]$. A shortest path from $u$ to $v$ in $AG_n$ is completely contained in $AG_n[I]$. That is, $d(u, v) = d_{AG_n[I]}(u, v)$.

(b) If $u_n = i$, $v_n = j$, and $v_1 = i$ for some $x \in \{1, 2\}$, then there exists a vertex $s \in V(AG_n[I])$ adjacent to $s$ such that $(u, s, v)$ is the minimum length path between $u$ and $v$ in $AG_n$, where $P$ is a path completely contained in $AG_n[I]$. That is, $d(u, v) = d_{AG_n[I]}(u, s) + 1 = d(u, v) + 1$.

(c) If $u_n = i$, $v_n = j$, and $v_1 = i$ for some $x \in \{3, 4, \ldots, n - 1\}$, then there exists a vertex $s \in V(AG_n[I])$ and $t \in V(AG_n[I])$, where $p$ is a path adjacent to $v$, and $(s, t) \in E_n[I]$ such that $d(u, p, s, t, v)$ is the minimum length path between $u$ and $v$ in $AG_n$, where $P$ is a path completely contained in $AG_n[I]$. That is, $d(u, v) = d_{AG_n[I]}(u, s) + 2 = d(u, v) + 2$.

**Example.** Suppose that $u$ and $v$ are two vertices in $AG_5$. If $u = 12345$ and $v = 21435$, then $u \in V(AG_5[5])$ and $v \in V(AG_5[5])$. A shortest path from $u$ to $v$ is $12345 \rightarrow 31245 \rightarrow 41235 \rightarrow 21435 \rightarrow 21345$, and case (a) holds. If $u = 12345$ and $v = 15432$, then $u \in V(AG_5[5])$ and $v \in V(AG_5[2])$. A shortest path from $u$ to $v$ is $12345 \rightarrow 31245 \rightarrow 41235 \rightarrow 21435 \rightarrow 21345 \rightarrow 15432$, and case (b) holds. If $u = 12345$ and $v = 34512$, then $u \in V(AG_5[5])$ and $v \in V(AG_5[2])$. A shortest path from $u$ to $v$ is $12345 \rightarrow 24315 \rightarrow 45312 \rightarrow 34512$, and case (c) holds.
3. PANPOSITIONAL HAMILTONICITY
OF THE ALTERNATING GROUP GRAPHS

In this section, we prove that the alternating group graph $AG_n$ is panpositionable hamiltonian for all $n \geq 3$. The basic idea is to study $AG_3$ and $AG_4$ first, and then to prove the result for $n \geq 5$ by induction on $n$.

Lemma 3. Alternating group graphs $AG_n$ are panpositionable hamiltonian for $n = 3, 4$.

Proof. For $n = 3$, since $AG_3$ is isomorphic to the complete graph $K_3$, the result holds for $n = 3$ trivially. Now, we consider the case $n = 4$. Let $u$ and $v$ be any two vertices of $AG_4$ in Figure 1.

The alternating group graph is known to be vertex symmetric and edge symmetric, and to have diameter $\lfloor \frac{3(n-2)}{2} \rfloor$ [7]. The diameter of $AG_4$ is 3. Without loss of generality, to prove this lemma it is enough to consider $u = 1234$ and $v = 3124$ for $(u, v) = 1$; $u = 1234$ and $v = 4321$ for $(u, v) = 2$; $u = 1234$ and $v = 2143$ for $(u, v) = 3$. For each $l \in \{d(u, v), d(u, v) + 1, \ldots, |V(AG_n)|/2\}$, we construct a hamiltonian cycle $HC$ of $AG_4$ such that $d_{HC}(u, v) = l$. These hamiltonian cycles $HC$ are listed below.

<table>
<thead>
<tr>
<th>$d(u, v)$</th>
<th>$d_{HC}(u, v)$</th>
<th>The cycle $HC$</th>
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<tr>
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<td>3</td>
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</tr>
<tr>
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<td>4</td>
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</tr>
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<tr>
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Thus the lemma holds.

In the following lemma, we show that there exist two vertex disjoint paths spanning all the vertices in an incomplete alternating group graph. We need the lemma later in our main theorem. One may skip the proof temporarily, and come back to it later.

Lemma 4. Suppose that $n \geq 5$, $I \subseteq \langle n \rangle$ with $|I| \geq 2$, $x_1 \in V(AG_n[I_1])$ and $x_2 \in V(AG_n[I_2])$ with $i_1 \neq i_2 \in I$. Then, for any pair of distinct vertices $(y_1, y_2)$ in $V(AG_n(I))$, there exist two disjoint paths, one joining $x_1$ and $y_1$ for some $i \in \{1, 2\}$, and the other joining $x_2$ and $y_2$ with $i \neq j$, such that these two paths span all the vertices in $AG_n(I)$.

Proof. Let $i_1, i_2, \ldots, i_{|I|}$ be $|I|$ distinct indices of $\langle n \rangle$. We prove this lemma by finding two disjoint paths $P_1$ and $P_2$ in $AG_n(I)$ such that $P_1$ joins $x_1$ and $y_1$, and $P_2$ joins $x_2$ and $y_2$ with $i \neq j$. Moreover, $P_1$ and $P_2$ span all the vertices in $AG_n(I)$. According to the location of $y_1$ and $y_2$, we have the following cases:

Case 1. Suppose that $y_1$ and $y_2$ are located in different subcomponents, and $x_1$ and $x_2$ are not both adjacent to $y_i$ for every $i \in \{1, 2\}$.

Subcase 1.1. Suppose that $x_1$, $x_2$, $y_1$, and $y_2$ are located in four different subcomponents. We assume that $y_i \in V(AG_n[I_3])$ and $y_j \in V(AG_n[I_4])$ with $|I| \geq 4$. See Figure 2a for an illustration. By Lemma 2, we can find a hamiltonian path $P_1$ from $x_1$ to $y_1$ in $AG_n((i_1, i_3))$. Similarly, we can find a hamiltonian path $P_2$ from $x_2$ to $y_2$ in $AG_n(I - \{i_1, i_3\})$. Therefore, $P_1$ and $P_2$ are two disjoint paths spanning all the vertices in $AG_n(I)$.

Subcase 1.2. Suppose that one of $y_1$, $y_2$ and one of $x_1$, $x_2$ are located in the same subcomponent. Without loss of generality, we may assume that $x_1$ and $y_1$ are located in the same subcomponent, $x_2$ and $y_2$ are located in different subcomponents, $y_1 \in V(AG_n[I_1])$, and $y_2 \in V(AG_n[I_3])$ with $|I| \geq 3$. See Figure 2b for an illustration. By Theorem 1, since $AG_n[I_1]$ is hamiltonian connected, we can find a hamiltonian path $P_1$ from $x_1$ to $y_1$ in $AG_n[I_1]$. By Lemma 2, we can
find a hamiltonian path $P_2$ from $x_2$ to $y_j$ in $AG_n(I - \{i_1\})$. Therefore, $P_1$ and $P_2$ are two disjoint paths spanning all the vertices in $AG_n(I)$.

**Subcase 1.3.** Suppose that $x_1$ and $y_i$ are located in the same subcomponent for some $i \in \{1, 2\}$, and $x_2$ and $y_j$ are located in the same subcomponent with $i \neq j$. We assume that $y_i \in V(AG_n[I_{i_1}])$ and $y_j \in V(AG_n[I_{i_2}])$ with $|I| \geq 2$. See Figure 2c for an illustration. Without loss of generality, we may assume that $i = 1$ and $j = 2$. By Theorem 1, since $AG_n[I_{i_1}]$ is hamiltonian connected, we can find a hamiltonian path $P_1$ from $y_1$ to $x_1$ in $AG_n[I_{i_1}]$. If $|I| \geq 3$, since $|N^*(y_2)| = 2$, we can find an edge $(y_2, y'_2) \in E_{n}^{ij}$ such that $y'_2 \in V(AG_n[I_{j}])$ for some $j \in I - \{i_1, i_2\}$. By Lemma 2, we can find a hamiltonian path $P'_2$ from $y'_2$ to $x_2$ in $AG_n(I - \{i_1\}) - \{y_2\}$. Let $P_2 = (y_2, y'_2, P'_2, x_2)$. If $|I| = 2$, by Theorem 1, there is a hamiltonian path $P'_2$ from $x_2$ to $y_j$ in $AG_n(I - \{i_2\})$. Therefore, $P_1$ and $P_2$ are two disjoint paths spanning all the vertices in $AG_n(I)$.

**FIG. 2.** Illustrations for Lemma 4.
y to x in AGn[t2]. Let P = (y2, P1, x2). Therefore, P1 and
P2 are two disjoint paths spanning all the vertices in AGn(I).

Case 2. Suppose that y1 and y2 are located in the same
subcomponent, and x1 and x2 are not both adjacent to yi for
every i ∈ {1, 2}.

Subcase 2.1. Suppose that y1, y2 ∈ V (AGn[i1]) or y1, y2 ∈
V (AGn[t2]) with |l| ≥ 2. See Figure 2d for an illustration.
Without loss of generality, we consider the former case and
assume that i = 1 and j = 2. By Theorem 1, AGn[i1] − {y2}
is hamiltonian connected, hence we can find a hamiltonian path
P1 from y1 to x1 in AGn[i1] − {y2}. If |l| ≥ 3, since
|N+(y2)| ≥ 2, we can find an edge (y2, y′2) ∈ Ei,j such that
y′2 ∈ V (AGn[j]) for some j ∈ {i1, i2, i3}. By Lemma 2, we
can find a hamiltonian path P′2 from y2 to x2 in AGn(I − {i1}).
If |l| = 2, there exists an edge (y2, y′2) ∈ Ei,j such that
y′2 ∈ V (AGn[t2]). By Theorem 1, there is a hamiltonian path
P′2 from y2 to x2 in AGn[t2]. Let P2 = (y2, y′2, P′2, x2). Therefore, the
paths P1 and P2 are two disjoint paths spanning all the vertices in
AGn(I).

Subcase 2.2. Suppose that y1, y2 ∈ V (AGn[i3]). Without
loss of generality, we consider two subcases:

Subcase 2.2.1. Suppose there exists some xi ∈ AS(y1) for
i ∈ {1, 2} with |l| ≥ 3. See Figure 2e for an illustration.
Without loss of generality, we assume that i = 1. Since
x1 ∈ AS(y1), we can find an edge (y1, y′1) ∈ Ei,j such that
y′1 ∈ V (AGn[i1]) and x1 ̸= y′1. By Theorem 1, we can find
a hamiltonian path P′1 from y1 to x1 in AGn[i1]. Let P1 =
(y1, y′1, P′1, x1). By Lemma 2, we can find a hamiltonian path
P′2 from y2 to x2 in AGn[I − {i1}]. If |l| ≥ 4, since
|N+(y′2)| ≥ 2, we can find an edge (y′2, y′′2) ∈ Ei,j such that
y′′2 ∈ V (AGn[j]) for some j ∈ {i1, i2, i3}. By Lemma 2, we
can find a hamiltonian path P′2 from y2 to x2 in AGn[I − {i1}].
Let P2 = (y2, y′2, y′′2, P′2, x2). Therefore, P1 and P2 are two disjoint paths spanning all the vertices in
AGn(I).

Subcase 2.2.2. Suppose that x1, x2 ̸∈ AS(y1) ∪ AS(y2) with
|l| ≥ 4. See Figure 2f for an illustration. Since |N+(y1)| = 2,
we can find an edge (y1, y′1) ∈ Ei,j such that y′1 ∈ V (AGn[1]) for
some j1 ∈ {i1, i2, i3}. By Lemma 2, we can find a hamiltonian path
P′1 from y1 to x1 in AGn[1]. Let P1 = (y1, y′1, P′1, x1). By Lemma 2, we
can find a hamiltonian path P′2 from x1 to x2 in AGn[I − {i1}]. If |l| ≥ 4,
since |N+(y′2)| ≥ 2, we can find an edge (y′2, y′′2) ∈ Ei,j such that
y′′2 ∈ V (AGn[j]) for some j2 ∈ {i1, i2, i3, i1}. By Lemma 2, we
can find a hamiltonian path P′2 from y2 to x2 in AGn(I − {i1, i2, i3, i1}). If |l| = 4, since |N+(y′2)| ≥ 2, we can
find an edge (y′2, y′′2) ∈ Ei,j such that y′′2 ∈ V (AGn[j]). Since
AGn[t2] is hamiltonian connected, there is a hamiltonian path
P′2 from y2 to x2 in AGn[t2]. Let P2 = (y2, y′2, y′′2, P′2, x2). Therefore, P1 and P2 are two disjoint paths spanning all the vertices in AGn(I).

Case 3. Suppose that x1 and x2 are adjacent to y2 for some
i ∈ {1, 2}. Without loss of generality, we assume that i = 1.
If y2 ∈ V (AGn[i1]), let P1 = {x1, y1, y2}. Suppose that |F| =
{x1, y1, y2}. By Lemma 2, we can find a hamiltonian path P2 from
x2 to y2 in AGn(I − F). If y2 ̸∈ V (AGn[i1]), let P1 = {x1, y2, y1, y2, P′2, x2}. Suppose that |F| = {x2, y1, y2}. By Lemma 2, we can find a hamiltonian path P2 from x2 to y2 in AGn(I − F). Therefore, P1 and
P2 are two disjoint paths spanning all the vertices in AGn(I).

Thus the lemma follows.

We now prove our main result.

Theorem 2. Alternating group graphs AGn are panposi-
tionable hamiltonian if n ≥ 3.

Proof. We prove this theorem by induction on n. By
Lemma 3, AG3 and AG4 are panpositionable hamiltonian.
Suppose that the result holds for AGn−1 for some n ≥ 5. We
observe that AGn can be recursively constructed from n copies
of AGn−1, and each AGn−1 is panpositionable hamiltonian by
the inductive hypothesis, for all n ≥ 5. Let s and t be two
distinct vertices of AGn. Then for each l ∈ [d(s, t), d(s, t) + 1, d(s, t) + 2, ..., |V (AGn)|], we shall find a hamiltonian cycle
of AGn such that the distance between s and t on the cycle is l.
The idea of the proof is to expand the path between s and t to various lengths by inserting one or more subcomponents
of AGn−1. We achieve this purpose by our expanding algorithm
described below, and we can construct a path connecting s
and t with the length of the path being l for any integer l with
d(s, t) ≤ l ≤ \frac{|V (AGn)|}{2}.

Case 1. Suppose that s and t belong to the same subcom-
ponent AGn[i]. There will be two subcases in Case 1; Figure 3a
and b illustrate Subcase 1.1 and Subcase 1.2, respectively.
See Figure 3a first. Suppose that s, t ∈ V (AGn[i]) for some
i ∈ [n]. By Proposition 3, d(s, t) = dAGn[i](s, t). Since AGn[i]
is isomorphic to AGn−1, by the inductive hypothesis, for each
l0 ∈ [d(s, t), d(s, t) + 1, d(s, t) + 2, ..., |V (AGn[i])| − d(s, t)],
we can construct a hamiltonian cycle HC[i] of AGn[i] such
that the distance between s and t on the cycle is l0. Let
u and v be the two neighbors of t on HC[i]. Let HC[i] =
{s, s, LP, u, t, v, RP, s}, and let P0 = ⟨s, LP, u, t, v, RP, s⟩. Without
loss of generality, let L(P0) = l0. By Proposition 2, d(t, u) = 1,
so we have |AS(t) ∩ AS(u)| = 1. This means that we can find a
subcomponent AGn[i] for which j1 ∈ (n − i), such that there exist two disjoint edges (u, p1) and (t, q1) in Ei,j1. By Proposition 1, (p1, q1) ∈ E(AGn[i]) and a vertex t′ ∈ V (AGn[h1]) such that (t, t′) ∈ Ei,j1. By Proposition 2,
\[d(t, v) \leq 2\] and hence \(AS(t) \supseteq \{j, h\}\), \(AS(t) \neq AS(v)\), and \(|N^*(v)| = 2\). So we can find another subcomponent \(AG_n([h])\) and a vertex \(v'\) \(\in V(AG_n([h]))\) such that \((v, v') \in E_n^{j,h}\) for some \(h \in (\eta) - \{i, j, h\}\). By Lemma 2, there exists a hamiltonian path \(HP = AG_n(n - \{i\})\) joining \(t'\) and \(v'\). Thus \((t, P_0, t', HP, v', v, RP, s)\) forms a hamiltonian cycle, and for each \(l_0 \in \{d(s, t), d(s, t) + 1, d(s, t) + 2, \ldots, |V(AG_n([i])| - d(s, t)\}, the distance between \(s\) and \(t\) on the cycle is \(l_0\).

Now we present an algorithm called st-expansion to insert one subcomponent of \(AG_n[i]\) into \(P_0\) as follows. Because \(p_s\) and \(q_s\) are adjacent, we may regard them as in the same subcomponent of \(AG_n[j]\), say \(C\). The subcomponent \(C\) is isomorphic to \(AG_n-2\). By Theorem 1, there is a hamiltonian path \(HP_s\) of \(C\) joining \(p_s\) and \(q_s\) with \(d(HP_s) = |V(AG_n-2)| - 1\), where \(m\) is the number of subcomponents of \(AG_n\) we wanted to insert. Thus, we can construct a path \(HP_s\) between \(p_s\) and \(q_s\), such that \(L(HP_s) = l_0|V(AG_n-2)| - 1\) for each integer \(l_0\) with \(1 \leq l_0 \leq n-1\). Let \(P_s = \{s, LP, u, p_s, HP_s, q_s, \ldots, q_1, t\}\). Thus we have \(L(P_s) = l_0 + l_1|V(AG_n-2)| = l_0 + l_{(n-2)}\). Since \(d(s, t) \leq l_0 \leq |V(AG_n([i])| - d(s, t), we have \(\frac{l_{(n-2)}}{2} + d(s, t) \leq L(P_1) \leq \frac{l_{(n-2)}}{2} + \frac{n-1}{2} - d(s, t)\). For each \(1 \leq l_1 \leq n-1\), \((\frac{l_{(n-2)}}{2} + \frac{n-1}{2} - d(s, t) \geq \frac{l_{(n-2)}}{2} + d(s, t)\) if \(n \geq 5\). Therefore, for each \(l_1 \in \{d(s, t), d(s, t) + 1, d(s, t) + 2, \ldots, (n-1)! - d(s, t)\}\), we can construct a path \(P_1\) from \(s\) to \(t\) such that the distance between \(s\) and \(t\) on the path is \(l_1\).

Similar as above, we can expand the path between \(s\) and \(t\) more. For each \(2 \leq x \leq \lfloor \frac{n}{2} \rfloor\), let \(u_{x-1}\) and \(t_{x-1}\) be two adjacent vertices on \(HP_{x-1}\), where \(HP_{x-1}\) is a hamiltonian path of \(AG_n[i]\) joining \(p_{x-1}\) and \(q_{x-1}\). By Propositions 1 and 2, there exist two distinct edges \((u_{x-1}, p_{x})\) and \((t_{x-1}, q_{x})\) in \(E_n^{(i-x,i)}\) for some \(j_x \in (\eta) - \{i, h, j\} \ldots, j\) such that \((p_x, q_x) \in E(AG_n([i]))\). See Figure 4c for an illustration. We can insert one subcomponent of \(AG_n[i]\) into \(P_0\) as follows. Because \(p_s\) and \(q_s\) are adjacent, we may regard them as in the same subcomponent of \(AG_n[j]\), say \(C\). The subcomponent \(C\) is isomorphic to \(AG_n-2\). By Theorem 1, there is a hamiltonian path \(HP_s\) of \(C\) joining \(p_s\) and \(q_s\) with \(L(HP_s) = |V(AG_n-2)| - 1\), where \(m\) is the number of subcomponents of \(AG_n[j]\) we wanted to insert.

Thus, we can construct a path \(HP_s\) between \(p_s\) and \(q_s\), such that \(L(HP_s) = l_0|V(AG_n-2)| - 1\) for each integer \(l_0\) with \(1 \leq l_0 \leq n-1\). Let \(P_s = \{s, LP, u, p_s, \ldots, q_s, \ldots, t\}\). Thus we have \(L(P_s) = l_0 + l_1|V(AG_n-2)| = l_0 + l_{(n-2)}\). Since \(d(s, t) \leq l_0 \leq |V(AG_n([i])| - d(s, t), we have \(\frac{l_{(n-2)}}{2} + d(s, t) \leq L(P_1) \leq \frac{l_{(n-2)}}{2} + \frac{n-1}{2} - d(s, t)\). For each \(1 \leq l_1 \leq n-1\), \((\frac{l_{(n-2)}}{2} + \frac{n-1}{2} - d(s, t) \geq \frac{l_{(n-2)}}{2} + d(s, t)\) if \(n \geq 5\). Therefore, for each \(l_1 \in \{d(s, t), d(s, t) + 1, d(s, t) + 2, \ldots, (n-1)! - d(s, t)\}\), we can construct a path \(P_1\) from \(s\) to \(t\) such that the distance between \(s\) and \(t\) on the path is \(l_1\).
can construct a path joining \( s \) and \( t \) with the length of the path being \( l \). We will use \( st\)-expansion for the remaining cases of the proof.

To construct a hamiltonian cycle, we consider the following two subcases:

**SUBCASE 1.1.** All the vertices of \( AG_n([j_1, \ldots, j_k]) \) are on the path \( P_x \), for some \( 1 \leq x \leq \lfloor \frac{n}{2} \rfloor \). See Figure 3a for an illustration. By Lemma 2, there is a hamiltonian path \( HP \) of \( AG_n(n - \{i, j_1, \ldots, j_k\}) \) joining \( t' \) and \( v' \) in which \( t' \in V(AG_n[h_i]) \) and \( v' \in V(AG_n[h_j]) \). Thus \( \langle s, P_x, t', HP, v', v, RP, s \rangle \) forms a hamiltonian cycle, and for each \( l \in \{d(s, t), d(s, t) + 1, d(s, t) + 2, \ldots, \lfloor \frac{|W(AG_n)|}{2} \} \), the distance between \( s \) and \( t \) on the cycle is \( l \).

**SUBCASE 1.2.** Not all the vertices of \( AG_n([j_1, \ldots, j_k]) \) are on the path \( P_x \), for some \( 1 \leq x \leq \lfloor \frac{n}{2} \rfloor \). See Figure 3b for an illustration. Then we can find two adjacent vertices \( y \) and \( z \) in \( AG_n[j_i] \), which are not on the path \( P_x \). Let \( F \subseteq V(P_x) \). By Proposition 1 and Proposition 2, there exist two distinct edges \( (y, y') \in E_n^{i,j_b} \) and \( (z, z') \in E_n^{j,b_i} \) such that \( y' \neq t' \in V(AG_n[h_i]) \) and \( z' \neq v' \in V(AG_n[h_j]) \), respectively. If \( AG_n[j_i] - F \) is isomorphic to \( AG_n[j_i - 1] \), by Theorem 1, there is a hamiltonian path \( HP \) from \( y \) to \( z \) in \( AG_n[j_i - 1] - F \). If \( AG_n[j_i] - F \) contains more than one subcomponent of \( AG_n[j_i] \), by Lemma 1 if \( n = 5 \), and by Lemma 2 if \( n > 5 \), there is a hamiltonian path \( HP \) from \( y \) to \( z \) in \( AG_n[j_i] - F \). By Lemma 4, there exist two disjoint paths \( DP_1 \) and \( DP_2 \), such that \( DP_1 \) joins \( t' \) and \( y' \), and \( DP_2 \) joins \( v' \) and \( z' \). Moreover, the two paths span all of the vertices in \( AG_n(n - \{i, j_1, \ldots, j_k\}) \). Thus \( \langle s, P_x, t', DP_1, y', HP, z', DP_2, v', v, RP, s \rangle \) forms a hamiltonian cycle, and for each \( l \in \{d(s, t), d(s, t) + 1, d(s, t) + 2, \ldots, \lfloor \frac{|W(AG_n)|}{2} \} \), the distance between \( s \) and \( t \) on the cycle is \( l \).

Now, we consider the case in which \( s \) and \( t \) belong to different subcomponents of \( AG_n \). Suppose that \( s \in V(AG_n[i]) \) and \( t \in V(AG_n[j]) \) for any \( i \neq j \in \{n\} \). By Proposition 3, there exists a minimum length path connecting \( s \) and \( t \) with the form \( \langle s, MP, t'' \rangle \) or \( \langle s, MP, t'' \rangle \), where \( MP \) is a path in \( AG_n[i] \), \( t'' \in V(AG_n[i]) \), and \( t' \in V(AG_n[j]) \). That is, \( d(s, t) = d_{AG_n[i]}(s, t'') + 1 = d(s, t'') + 1 \) or \( d(s, t) = d_{AG_n[i]}(s, t'') + 2 = d(s, t'') + 2 \). Thus we have the following cases:

**CASE 2.** Suppose that \( s \) and \( t \) belong to different subcomponents of \( AG_n \), and the minimum length path connecting \( s \) and \( t \) has the form \( \langle s, MP, t'' \rangle \). Then \( d(s, t) = d(s, t'') + 1 \). See Figure 5a for an illustration. Since \( AG_n[i] \) is isomorphic to \( AG_n[i - 1] \), by the inductive hypothesis, for each \( l_0 \in \{d(s, t''), d(s, t'') + 1, d(s, t'') + 2, \ldots, \lfloor |W(AG_n[i])| - d(s, t'') \} \), we can construct a hamiltonian cycle \( HC_i \) of \( AG_n[i] \) such that the distance between \( s \) and \( t' \) on the cycle is \( l_0 \).
u and v be the two neighbors of \(t''\) on HC\(_i\). Let HC\(_i\) = (s, LP, u, t''', v, RP, s), and let P\(_0\) = (s, LP, u, t'', t). Without loss of generality, let \(L(P_0) = l_0 + 1\). By Proposition 2, d(t'', u) = 1, so we have \(|AS(t'') \cap AS(u)| = 1\). This means that we can find a subcomponent AG\(_n\)[j\(_i\)] in which j\(_i\) \(\in\) (n) - {i}. If t \(\notin\) V (AG\(_n\)[j\(_i\)]), by using st''-expansion, the proof is the same as Case 1 but we replace vertex t in Case 1 with vertex t'' in this case. So we consider the case in which t \(\in\) V (AG\(_n\)[j\(_i\)]), that is, j\(_i\) = j. Let q\(_1\) = t. There exist two disjoint edges (u, p\(_1\)) and (t'', q\(_1\)) in E\(_n\)\(_j\). By Proposition 1, (p\(_1\), q\(_1\)) \(\in\) E(AG\(_n\)[j\(_i\)]). By Proposition 2, d(t'', v) \(\le\) 2 hence AS(t'') = {j\(_i\)}, and AS(t'') \(\neq\) AS(v). Since |N\(^s\)(t'')| = 2, we can find a subcomponent AG\(_n\)[h\(_i\)] and a vertex t' \(\in\) V (AG\(_n\)[h\(_i\)]) such that (t', t'') \(\in\) E\(_n\)\(_j\) for some h\(_i\) \(\in\) (n) - {i, j\(_i\)}. Since |N\(^s\)(v)| = 2 and AS(t'') \(\neq\) AS(v), we can find a subcomponent AG\(_n\)[h\(_j\)] and a vertex v' \(\in\) V (AG\(_n\)[h\(_j\)]) such that (v, v') \(\in\) E\(_n\)\(_j\) for some h\(_j\) \(\in\) (n) - {i, j, j\(_i\)}. By Lemma 2, there exists a hamiltonian path HP of AG\(_n\)((n) - {i}) joining t and v'. Thus (s, P\(_0\), t, HP, v', v, RP, s) forms a hamiltonian cycle, and for each l \(\in\) |d(s, t), d(s, t) + 1, d(s, t) + 2, ..., |V(AG\(_n\)[i])| - d(s, t) + 1|, the distance between s and t on the cycle is l.

**SUBCASE 2.2.** Not all the vertices of AG\(_n\)(j\(_1\), ..., j\(_k\)) are on the path P\(_x\) for some 1 \(\le\) x \(\le\) \([\frac{n}{2}]\). See Figure 5a for an illustration. Then, we can find two adjacent vertices y and z in AG\(_n\)(j\(_x\)), which are not on the path P\(_x\). Let F \(\subseteq\) V (P\(_x\)). By Proposition 1 and Proposition 2, there exist two distinct edges (y, y') \(\in\) E\(_n\)\(_j\) and (z, z') \(\in\) E\(_n\)\(_j\) such that y' \(\neq\) t' \(\in\) V (AG\(_n\)[h\(_i\)]) and z' \(\neq\) v' \(\in\) V (AG\(_n\)[h\(_j\)]), respectively. If AG\(_n\)[j\(_x\)] - F is isomorphic to AG\(_n\)-2, by Theorem 1, there is a hamiltonian path HP from y to z in AG\(_n\)[j\(_x\)] - F. If AG\(_n\)[j\(_x\)] - F contains more than one subcomponent of AG\(_n\)[j\(_x\)], by Lemma 1 if n = 5, and by Lemma 2 if n > 5, there is a hamiltonian path HP from y to z in AG\(_n\)[j\(_x\)] - F. By Lemma 4, there exist two disjoint paths DP\(_1\) and DP\(_2\), such that DP\(_1\) joins t' and y', and DP\(_2\) joins v' and z'. Moreover, the two paths span all of the vertices in AG\(_n\)((n) - {i, j, j\(_i\), j\(_x\)})). Thus (s, P\(_x\), t, t', t'', t''', DP\(_1\), y', y, HP, z, z', DP\(_2\), v', v, RP, s) forms a hamiltonian cycle, and for each l \(\in\) |d(s, t), d(s, t) + 1, d(s, t) + 2, ..., |V(AG\(_n\)[i])| - d(s, t') + 1|, the distance between s and t on the cycle is l.

**CASE 3.** Suppose that s and t belong to different subcomponents of AG\(_n\), and the minimum length path connecting s and t has the form (s, MP, t', t', t). Then d(s, t) = d(s, t') + 2. See Figure 5b for an illustration. Since AG\(_n\)[i] is isomorphic to AG\(_n\)-1, by the inductive hypothesis, for each l \(\in\) |d(s, t'), d(s, t') + 1, d(s, t') + 2, ..., |V(AG\(_n\)[i])| - d(s, t')|, we can construct a hamiltonian cycle HC\(_i\) of AG\(_n\)[i] such that the distance between s and t on the cycle is l. Let HC\(_i\) = (s, LP, u, t''', v, RP, s) forms a hamiltonian cycle, and for each l \(\in\) |d(s, t), d(s, t) + 1, d(s, t) + 2, ..., |V(AG\(_n\)[i])| - d(s, t')|, the distance between s and t on the cycle is l.

FIG. 5. Theorem 2, Case 2 and Case 3.
Lemma 2, there is a hamiltonian path joining $s$ and $t$. By Proposition 2, $d(t'', u) = 1$, so we have $|AS(t'') \cap AS(u)| = 1$. This means that we can find a subcomponent $AG_n[j_1]$ for which $j_1 \in \{n\} - \{i\}$. If $t'' \notin V(AG_n[j_1])$, by using st'-expansion, the proof is the same as Case 1 but we replace vertex $t$ in Case 1 with vertex $t''$ in this case. So we consider the case in which $t, t'' \in V(AG_n[j_1])$, that is, $j_1 = j$. There exist two disjoint edges $(u, p_1)$ and $(t'', t')$ in $E_n[j_1]$. By Proposition 1, $(p_1, t') \notin V(AG_n[j_1])$. By Proposition 2, $d(t'', v') \leq 2$ hence $AS(t'') = \{j_1\}$, and $AS(t'') \neq AS(v')$. Since $|N^+(t'')| = 2$, we can find a subcomponent $AG_n[h_i]$, and a vertex $t'' \in V(AG_n[h_i])$ such that $(t'', t'') \in E_n[h_i]$ for some $h_i \in \{n\} - \{i, j_1\}$. Since $|N^+(v')| = 2$ and $AS(t'') \neq AS(v')$, we can find a subcomponent $AG_n[h_i]$, and a vertex $v' \in V(AG_n[h_i])$ such that $(v, v') \in E_n[h_i]$ for some $h_i \in \{n\} - \{i, j_1, t''\}$. Let $F \subseteq V(AG_n)$ and $F' = \{t''\}$. By Lemma 2, there exists a hamiltonian path $HP_1$ of $AG_n(n - \{i\})$ joining $t$ and $v'$. Thus $(s, P_0, t', HP, v', v, RP, s)$ forms a hamiltonian cycle, and for each $l_0 \in \{d(s, t), d(t, s)\}$, $|\{d(s, t) + 2, \ldots, |V(AG_n)| - d(s, t)\}| = (d(s, t) + 1)$, the distance between $s$ and $t$ on the cycle is $l_0$. Now we modify the st'-expansion slightly to expand the path $P_0$ between $s$ and $t$ to various lengths. We describe the details as follows.

Suppose that $n \geq 6$. See Figure 4d for an illustration. We can insert one subcomponent of $AG_n[j_1]$, which is isomorphic to $AG_{n-2}$, into $P_0$ as follows. Because $d(p_1, t) = 2$, which is less than the diameter of $AG_{n-2}$, and by the symmetric property of the alternating group graph, we may regard $p_1$ and $t$ as in the same subcomponent of $AG_{n-2}$, say $C$. By Lemma 2, there is a hamiltonian path $HP_1$ of $C - F_j$ joining $p_1$ and $t$ with $L(HP_1) = |V(AG_n) - 2|$. Let $C'$ be the $m$ subcomponents of $AG_n[j_1]$ we wanted to insert into $P_0$, where $m$ is the number of subcomponents of $AG_n[j_1]$. We regard $p_1$ and $t$ as in different subcomponents of $AG_n[j_1]$. By Lemma 2, there is a hamiltonian path $HP_1$ of $C' - F_j$ joining $p_1$ and $t$ with $L(HP_1) = |mV(AG_n) - 2|$. Thus, we can construct a path $HP_1$ between $p_1$ and $t$ such that $L(HP_1) = |V(AG_n) - 2| - 2l_0 = \frac{L(AG_n)}{2} - 2$. Since $d(s, t) = 2 \leq l_0 \leq |V(AG_n)| - d(s, t) + 2$, we have $L(AG_n) \leq \frac{L(AG_n)}{2} + \frac{L(AG_n)}{2} - 2$. If $n \geq 6$, then for each $l_0 \leq n - 1$, we have $L(AG_n) \leq \frac{L(AG_n)}{2} + \frac{L(AG_n)}{2} - 2$. Hence we can construct a path $P_1$ from $s$ to $t$ such that the distance between $s$ and $t$ on the path is $l_1$. Then, similar to the st'-expansion described in Case 1, we can expand the path between $s$ and $t$ such that for each $l_0 \in \{d(s, t), d(s, t) + 1, d(s, t) + 2, \ldots, \frac{n+1(n-1)}{2} - d(s, t)\}$, we can construct a path $P_2$ from $s$ to $t$ such that the distance between $s$ and $t$ on the path is $l_0$. Hence for any integer $l$, with $d(s, t) \leq l \leq |V(AG_n)|$, we can construct a path joining $s$ and $t$ with the length of the path being $l$.

For $n = 5$, that is, $AG_5$, we have $d(s, t) = 4$ in this case. As described above, $(s, LP, u, t', v, RP, s)$ forms a hamiltonian cycle, and for each $l_0 \in \{4, 5, 6, \ldots, 12\}$, the distance between $s$ and $t$ on the cycle is $l_0$. Let $F_2 \subseteq V(AG_n[j_1])$ and $F_j = \{t'\}$. By Theorem 1, we can find a hamiltonian path $HP_1$ of $AG_n[j_1]$ - $F_j$ joining $p_1$ and $t$. Let $P_1 = (s, LP, u, p_1, HP_1, t)$. We have $11 \leq L(P_1) \leq 19$. Therefore, for each $l_1 \in \{4, 5, 6, \ldots, 19\}$, we can construct a path $P_2$ from $s$ to $t$ such that the distance between $s$ and $t$ on the path is $l_1$. Let $u_1$ and $t_1$ be two adjacent vertices on $HP_1$. Then, for each $l_2 \in \{4, 5, 6, \ldots, 19\}$, we can construct a path $P_2$ from $s$ to $t$ such that the distance between $s$ and $t$ on the path is $l_2$.

To construct a hamiltonian cycle, the proof is the same as that given in Subcase 2.1 and Subcase 2.2 by replacing vertex $t'$ in Case 2 with vertex $t''$ in this case.

Hence the theorem is proved.

4. CONCLUDING REMARKS

In this article, we have proposed a new concept called panpositionable hamiltonicity. We now explain some relationship between panpositionable hamiltonicity and panconnectivity. A hamiltonian graph $G$ is panpositionable if for any two different vertices $x$ and $y$ of $G$ and for any integer $l$ satisfying $d(x, y) \leq l \leq |V(G)| - d(x, y)$, there exists a hamiltonian cycle $C$ of $G$ such that the relative distance between $x$, $y$ on $C$ is $l$. A graph $G$ is panconnected if there exists a path of length $l$ joining any two different vertices $x$ and $y$ with $d(x, y) \leq l \leq |V(G)| - 1$. If $G$ is panpositionable, it is clear that there exists a path of length $l$ joining any two different vertices $x$ and $y$ with $|V(G)| - d(x, y) + 1 \leq l \leq |V(G)| - 1$, then we can conclude that $G$ is panconnected. By Theorem 1, the fault tolerant hamiltonian properties of the alternating group graph $AG_n$, there exists a path of length $l$ joining any two different vertices $x$ and $y$ with $\frac{n^2}{2} - 4 \leq l \leq \frac{n^2}{2} - 1$ in $AG_n$ if $n \geq 4$. Therefore, we can obtain the following known result as a corollary.

**Corollary 1.** [1] Alternating group graphs $AG_n$ are panconnected for all $n \geq 4$.

We give an example to show that a panconnected graph $G$ is not necessarily panpositionable hamiltonian. Let $n, s_1, s_2, \ldots, s_r$ be integers with $1 \leq s_1 < s_2 < \cdots < s_r$. The circulant graph $C(n; s_1, s_2, \ldots, s_r)$ is a graph with vertex set $\{0, 1, \ldots, n - 1\}$. Two vertices $i$ and $j$ are adjacent if and only if $i - j = \pm s_k$ (mod $n$) for some $k$ where $1 \leq k \leq r$. We can check that $C(n; 1, 2)$ is panconnected by brute force for $n \in \{5, 6, 7, 8, 9, 10\}$. However, $C(10; 1, 2)$ is not panpositionable hamiltonian. Figure 6 shows the structure of $C(10; 1, 2)$. Consider vertex 0 and vertex 2, with $d(0, 2) = 1$. We prove by contradiction that $C(10; 1, 2)$ does not contain a hamiltonian cycle $HC$ with $d_{HC}(0, 2) = 5$. Suppose to the contrary that $HC$ is a hamiltonian cycle.
of $C(10; 1, 2)$ with $d_{HC}(0, 2) = 5$. There are three possible paths, $P_1 = (0, 8, 9, 1, 3, 2)$, $P_2 = (0, 9, 1, 3, 4, 2)$, and $P_3 = (0, 1, 3, 5, 4, 2)$, of length 5 joining vertex 0 and vertex 2. If $HC$ contains $P_1$, then the edges $(0, 1), (0, 2), (0, 9)$ cannot belong to $HC$. If $HC$ contains $P_2$ or $P_3$, then the edges $(2, 0), (2, 1), (2, 3)$ cannot belong to $HC$. Hence for $n = 10$, there does not exist any hamiltonian cycle in $C(10; 1, 2)$ such that the distance on the cycle between vertex 0 and vertex 2 is 5. So $C(10; 1, 2)$ is not panpositionable hamiltonian. In fact, the circulant graph $C(n; 1, 2)$ is pan-connected for every $n \geq 5$, but it is not panpositionable hamiltonian for some values of $n$. Therefore, the panpositionable hamiltonian property is a stronger property for an interconnection network. Future work will try to find the panpositionable hamiltonicity of other interconnection networks and some relationships between these hamiltonian-like concepts.

**Acknowledgments**

The authors thank the anonymous referees for their helpful suggestions and comments which, remove some ambiguities and improve presentation.

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